Spanning Forests on Random Planar Lattices

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Dedicated to Édouard Brézin and Giorgio Parisi,
pioneers also in this subject,
on the occasion of their special birthday.

Abstract

The generating function for spanning forests on a lattice is related to the q-state Potts model in a certain $q \to 0$ limit, and extends the analogous notion for spanning trees, or dense self-avoiding branched polymers. Recent works have found a combinatorial perturbative equivalence also with the (quadratic action) $O(n)$ model in the limit $n \to -1$, the expansion parameter t counting the number of components in the forest.
We give a random-matrix formulation of this model on the ensemble of degree-k random planar lattices. For $k = 3$, a correspondence is found with the Kostov solution of the loop-gas problem, which arise as a reformulation of the (logarithmic action) $O(n)$ model, at $n = -2$.
Then, we show how to perform an expansion around the $t = 0$ theory. In the thermodynamic limit, at any order in t we have a finite sum of finite-dimensional Cauchy integrals. The leading contribution comes from a peculiar class of terms, for which a resummation can be performed exactly.

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1 Introduction

The $O(n)$-invariant σ-model for $n = 0, -1, -2$ defined on a generic graph G has a very interesting combinatorial interpretation.

Already in early 70's, it has been observed that the n-vector model in the limit in which $n \to -2$ is equivalent to a free fermionic theory [1] . But the quadratic (Gaussian) term is the Laplacian on the graph G and the partition function is its determinant, which according to Kirchhoff matrix-tree theorem, provides, once the zero-mode has been removed, the weight of spanning trees of the graph G [2].

In 1980 Parisi and Sourlas [3] showed the equivalence of the n-vector model in a limit in which $n \to 0$, which was already known to describe the critical behaviour of polymers [4], with a supersymmetric $\mathfrak{osp}(2|2)$ model in which the loops in Feynman graphs, which vanish in the $n \to 0$ limit, gives zero contribution, because of the cancellation between bosons and fermions (independently also McKane noticed that fermions can be used to cancel the contribution of bosonic loops [5]). Also this construction is independent from the choice of the graph G, as only exploits the symmetry properties in the target space. The choice for the $O(n)$-invariant

model as a σ-model provides exactly the partition function of self-avoiding walks on the graph (see for example [6]).

More recently, we have shown [7] that the generating function of spanning forests in a graph G can be represented as a Grassmann integral of the exponential of a local fermionic weight involving a Gaussian term together with a special nearest-neighbour four-fermion interaction (see also [8]). Furthermore, the fermionic model possesses a hidden $\mathfrak{osp}(1|2)$ supersymmetry non-linearly realised. In [7] we also discussed briefly how this fermionic model can be mapped, at least in perturbation theory, onto an $\mathfrak{osp}(1|2)$-invariant σ-model with spins taking values in the unit supersphere in $\mathbb{R}^{1|2}$, or an $O(n)$-invariant σ-model with spins taking values in the unit sphere in $\mathbb{R}^n$ (also known as *n-vector model*), analytically continued to $n = -1$. The parameter t which appears in the generating function of the forests to count the number of trees in a forest is related to the coupling constant in the σ-model (with an important inversion of sign). It is remarkable that the same generating function can be obtained by a suitable limit $q \to 0$ of the q-state Potts model defined on the same graph G (see for example [9]).

Very detailed information on these models can be obtained by considering *regular* graphs, and in particular in two dimensions, where methods of Conformal Field Theory and Integrable Systems apply.

The critical exponents of the n-vector model, at least on the range $n \in [-2, 2]$, can be computed *exactly*, thanks to a mapping onto the solid-on-solid model [10] of a suitable choice of the weights, which, in the high-temperature expansion forbids loop crossing. We shall call this variant of the n-vector model the *Nienhuis model*, or *Loop-gas model* (as it is a 'hard-core lattice gas' in which the elementary objects are self-avoiding loops). Also the critical behaviour of the Potts model can be analyzed exactly, thanks to the mapping onto an ice-type model, which has been constructed both algebrically [11] and combinatorially [12]. This is once more mapped to a solid-on-solid model [13]. And the critical limit of the solid-on-solid model is recovered by using the Coulomb-gas picture [14]. More precisely, the critical behaviour of this n-vector model is given by a conformal field theory (CFT) with central charge

$$c(n) \;=\; 1 - \frac{6}{m(m+1)} \tag{1.1}$$

where the parameter m is related to n by the relation

$$n \;=\; 2\cos\frac{\pi}{m} \tag{1.2}$$

and to the Coulomb-gas coupling constant by

$$g_0 \;=\; 1 + \frac{1}{m}\,. \tag{1.3}$$

Please, remark that $c(-1) = -3/5$ and $c(-2) = -2$. This means that the Nienhuis model does not describe at $n = -1$ the universality class of spanning forests. Indeed, a direct perturbative analysis of the $O(n)$ σ-model on a square lattice [15] or on a triangular lattice [16] at $n = -1$ shows that the model is asymptotically free for $t = 0^+$. As a result, the ultra-violet fixed point is the free theory which describes trees, therefore, aside logarithmic violations, the central charge is $c = -2$.

Beyond the study of a model on a fixed periodic planar graph, a lot of progress has been achieved by considering an ensemble of planar graphs [17,18]. Such a study has both an interest *per se* of combinatorial nature, and a relevance in connection to the original case of regular graphs. Indeed, after the work of Knizhnik, Polyakov and Zamolodchikov [19, 20], it is now understood that, for systems showing conformal invariance at criticality, statistical averages in the two ensembles have related critical behaviours, so that informations in one context can be inferred from the other, e.g. concerning the critical exponents, the conformal families of operators, and their dimensions. [19–23] In many cases, cross-checks of these predictions have been performed [23–25].

A deep understanding of KPZ relation is a hard and active field [21, 26, 27]. However, some heuristic reasons can be given at least for the existence of a relation of this kind. In two dimensions, at criticality, scale invariance, combined with (discretized) Euclidean symmetries, is promoted to the symmetry described by the full Virasoro Algebra. Analogously, statistical mechanics models on random planar graphs, when reaching simultaneously the large-volume limit and the critical point for the "matter fields" (*double scaling limit*), show the scale invariance pertinent to criticality, combined with the (discretized) invariance under local diffeomorphisms (as bare random planar graphs describe a discretization of two-dimensional quantum gravity). But a conformal theory in presence of two-dimensional quantum gravity enjoys a symmetry corresponding to a $\mathrm{SL}(2, \mathbb{R})$ current algebra [19], which is larger than that described by the Virasoro algebra.

This richer structure is in a way at the root of the fact that many results exist for the apparently harder counting problem on random planar graphs (which involves a double average), and still lacks in the Euclidean case (and indeed provide a hint to critical aspect of these quantities, through KPZ). In words more appropriate to the discrete setting, in many combinatorial approaches (among which the present paper), the interplay between degrees of freedom of the lattice and of matter fields plays a crucial role in simplifying the expressions. On the other side, many Euclidean concepts involving a natural notion of distance are harder to define cleanly in the random-graph setting (and, if defined through geodesic distance, hard to compute).

On the other side, the generating function of statistical configurations over random graphs (as well as many other "global" physical observables, such as susceptibilities) can be written as the Feynman expansion of a proper action of a zero-dimensional field theory: the replacement of real or complex bosonic fields with $N \times N$ symmetric- or hermitean-matrix fields allows to count graphs of genus $\mathfrak{h}$ with a weight proportional to $N^{-2\mathfrak{h}}$, and a large-N limit, achieved via steepest descent or continuous approximation of matrix spectra, gives the restriction to planar graphs. Such a strategy, started with the seminal works of [17, 18], had a strong development in the subsequent thirty years, and now deserves the name of *Random Matrix* technique (see [28] for a recent pedagogical introduction).

Indeed, after the Ising [29] and Potts [30] models, also the Nienhuis model has been solved [31, 32] on random planar graphs, and deeply studied (see for example [33–35]). In [36], as we are interested in the cases $n = -1, -2$ of the Nienhuis model, a combinatorial reformulation of these problems has been introduced to achieve the random matrix solution with no need of an analytical continuation.

A detailed account of these different research areas when they overlap on geometrical critical phenomena is given by Duplantier and Kostov [25].

But the model of spanning forests in an ensemble of random graphs escapes the realm of exact results. Only the limit of spanning trees, that is $t = 0$, that we have seen corresponds to the Nienhuis model at $n = -2$, has been studied, and for regular graphs with coordination number 3. For spanning forests it is necessary to dispose of informations at finite values of the coupling constant, at least in perturbation theory.

A full discussion on the contents of this paper is postponed to the end of section 2, after that some other definitions are introduced.

2 The model

Since here on, and according to common use in Random Matrix, we call a "graph" G (or, more precisely, a "fatgraph"), what is indeed an orientable 2-dimensional cell complex, i.e. G is determined not only by the sets $V(G)$ and $E(G)$, respectively for vertices and edges, but also by a consistent choice of the set $F(G)$ for the *faces* (or, equivalently, for each vertex, the outgoing edges have a given cyclic ordering). For this reason, the *genus* $\mathfrak{h}$ of G is univocally and easily determined, for example via Euler formula (V, E, F and K denote respectively the

number of vertices, edges, faces and connected components in the cell complex)

$$2\mathfrak{h} = 2K + E - V - F\,. \tag{2.1}$$

Consider the ensemble of all connected graphs with vertices of degree k, with a measure depending from the genus (and allowing to take a "planar" limit). Most commonly studied cases are $k = 3$ of $k = 4$: we will study the generic case, but with special attention to $k = 3$, both because it is the ensemble studied in Kostov solution, and therefore this allows for a direct comparison of results, and because the generating function $A_1(\omega)$ for cubic trees is the one with the simplest explicit formula.

For a connected graph G with V vertices and genus $\mathfrak{h}$ we consider the customary (unnormalized) measure for Random Matrix theory

$$\mu_{g,N}(G) = \frac{1}{|\mathrm{Aut}(G)|} g^V N^{-2\mathfrak{h}}\,, \tag{2.2}$$

and the generating function (for connected graphs) is obtained by the Random Matrix technique with a single matrix field:

$$Z_0 = \sum_G \mu_{g,N}(G) = \frac{1}{N^2} \ln \int_{N\times N} \mathrm{d}M\, e^{N\mathrm{tr}\left(-\frac{1}{2}M^2 + \frac{g}{k}M^k\right)}\,. \tag{2.3}$$

Here the integral is over a set of N^2 real variables, arranged in the Hermitian matrix M, and must be intended as the formal Feynman expansion in the parameter g [28]. Indeed, a sketch of the technique is the following: the Feynman diagrammatics leads to the generating function of all connected graphs, and the traces due to the "matrix of fields", jointly with the Wick rule $\langle M_{ij} M_{\ell k}\rangle = \delta_{j\ell}\delta_{ki}$, lead to a combinatorics on face-indexing that, via Euler formula, allows to count the genus of the graph with the desired factor $N^{-2\mathfrak{h}(G)}$.

This is a *one-matrix theory*, i.e. we have only one matrix of fields. We recall here some results. First perform the change of variables $M \to U\Lambda U^{-1}$, with U unitary and Λ diagonal, the Jacobian corresponding to the square Vandermonde determinant $\Delta^2(\vec{\lambda}) = \prod_{i\neq j} |\lambda_i - \lambda_j|$. Angular degrees of freedom do not appear in the action, and are trivially integrated, similarly the ordering of the eigenvalues is irrelevant, and is trivially summed over, and the partition function reads

$$Z_0 \propto \frac{1}{N^2} \ln \int \mathrm{d}^N\vec{\lambda}\, \Delta^2(\vec{\lambda})\, \exp\left[N \sum_i \left(-\tfrac{1}{2}\lambda_i^2 + \tfrac{g}{k}\lambda_i^k\right)\right]. \tag{2.4}$$

At this point, many tools allow to extract the asymptotic behaviour in V in the planar limit (saddle point, orthogonal polynomials, loop equations...). The result for the leading behaviour near to the radius of convergence g_c of the series is, for any genus $\mathfrak{h}$,

$$Z_0(g, \mathfrak{h}) \sim c(\mathfrak{h}) \sum_V \left(\frac{g}{g_c}\right)^V V^{-3+\gamma+\mathfrak{h}\gamma'} N^{-2\mathfrak{h}}\,, \qquad \gamma = -\frac{1}{2}\,; \qquad \gamma' = \frac{5}{2}\,. \tag{2.5}$$

The quantity in (2.4) is the *pure gravity* partition function, which should be adopted as the appropriate normalization factor when comparing with the analogous expression in the case of a matrix theory describing an ensemble of combinatorial structures on the graph (interpreted as *matter fields* coupled to the gravity). In our case, we should deal with Fortuin-Kasteleyn random clusters for the analytic continuation in q of the q-colour Potts Model, in a limit $q \to 0$ corresponding to a restriction to forests, with a factor t per tree in the forest.

So, let $\mathcal{F}(G)$ be the set of spanning forests over the graph G. Say for short that $F \prec G$ if $F \in \mathcal{F}(G)$, and call $K(F)$ the number of its connected components. Given two graphs G, H with $H \subseteq G$, define G/H the *contraction* of G by H, i.e. the graph in which vertices i and j are identified if an edge (ij) is in $E(H)$ (this is the concept used, for example, in deletion/contraction operations for Tutte-Grothendieck invariants). In particular, if G has V vertices and E edges, and $F \prec G$ has E' edges, the graph G/F has $V - E'$ vertices and $E - E'$

edges. Furthermore, as any tree is homotopic to a point, G and G/F have the same topology ($\mathfrak{h}(G) = \mathfrak{h}(G/F)$), however they may have different automorphism groups.

At fixed graph G, we define the spanning-forest generating function as

$$Z(t; G) = \sum_{F \prec G} \mu_{t,G}(F) \,, \tag{2.6}$$

through the (unnormalized) measure

$$\mu_{t,G}(F) = t^{K(F)} \frac{|\mathrm{Aut}(G)|}{|\mathrm{Aut}(G/F)|} \,. \tag{2.7}$$

The symmetry factor, depending on the pair (G, F), has been introduced for later convenience. Here we remark that, if we worked with (leg-)labeled graphs, and exponential generating function, i.e.

$$Z = \sum_{G \text{ labeled}} \frac{1}{(2|E(G)|)!} \mu(G) \,, \tag{2.8}$$

the rephrasing of the measure above would involve no factors whatsoever. Furthermore, for large graphs (which are the dominant class in the thermodynamic limit), these symmetry factors are almost surely 1.

A remarkable exception to this last argument occurs in our exact resummation of section 9, where the contributions of certain families of "large-volume" graphs are resummed into the evaluation of certain integrals over graphs with size of order 1, for which the inclusion of the proper symmetry factor is important.

We are interested in the macrocanonical average in the random lattice ensemble, mainly in the limits $N \to \infty$ (*planar* limit), and $g \to g_c(t)$, the radius of convergence of the resulting series (*thermodynamic* limit). A key fact is that interchanging the two sum operations (over graphs G, and over forests F on G) the expression largely simplifies. Indeed, easy manipulations give

$$\begin{aligned} Z(t, g, N) &= \sum_G \mu_{g,N}(G) Z(t; G) = \sum_G \sum_{F \prec G} \mu_{g,N}(G) \mu_{t,G}(F) \\ &= \sum_G g^{|V(G)|} N^{-2\mathfrak{h}(G)} \sum_{F \prec G} \frac{t^{K(F)}}{|\mathrm{Aut}(G/F)|} \\ &= \sum_F t^{K(F)} g^{|V(F)|} \sum_{G \succ F} \frac{N^{-2\mathfrak{h}(G/F)}}{|\mathrm{Aut}(G/F)|} \,, \end{aligned} \tag{2.9}$$

and in particular, for the planar case $N \to \infty$, we have $N^{-2\mathfrak{h}(G)} \to \delta_{\mathfrak{h}(G),0}$, and

$$Z(t, g) = \sum_F t^{K(F)} g^{|V(F)|} \sum_{\substack{G \succ F \\ \text{planar}}} \frac{1}{|\mathrm{Aut}(G/F)|} \,. \tag{2.10}$$

Estimating this function as well as possible, in the neighbourhood of $t \sim 0$ and $g \to g_c(t)$, that is, for graphs in the thermodynamic limit, and near to the critical point of spanning trees, is the main goal of this paper.

In section 3 we give some preliminary results of combinatorial nature. In section 4 we describe a standard random-matrix approach to the problem at generic t, g and N. In sections 5 and 6, it is shown a correspondence with the $O(n)$ loop-gas model.

In sections 7–9, the function (2.10) is shown to admit a diagrammatic perturbative expansion in the parameter t, i.e. the coefficients $Z_n(g)$ of the series

$$Z(t, g) = \sum_{n \geq 1} t^n Z_n(g) \tag{2.11}$$

can be calculated with a combinatorial technique, each term being a finite sum of finite-dimensional integrals.

The way in which these diagrams are effectively evaluated is described in section 10, while sections 11 and 12 describe how to extract the leading behaviour from this sum, in the double limit $t \to 0$ and $g \nearrow g_c$, for $t/\ln(g_c/g)$ below a critical threshold.

While sections 4, 5 and 6 involve concepts of Random Matrix Theory, all other sections are purely of combinatorial nature, and use elementary self-contained methods (except for the isolated appearence of the non-trivial Lévy generalized Central Limit Theorem, in section 8, which furthermore is not mandatory in the logic of the paper, as the results are rederived in section 10).

3 Preliminary combinatorial results

Here we make a short review of results for certain counting problems which will arise in our analysis. This section does not make use of any tool from Random Matrix theory, and is fully elementary and self-contained. We call $\mathbb{H}$ the upper half plane, and $\mathbb{D}$ the unit disk.

Problem 1. *Counting the configurations of n non-intersecting arcs in $\mathbb{H}$, with endpoints in $\partial\mathbb{H}$ (cfr. fig. 1, top left).*

Such a configuration is called a *link pattern* (in $\mathbb{H}$). Their number is C_n, the n-th Catalan number. We adopt the convention $C_0 = 1$ for the empty configuration. Call $C(q) = \sum_n C_n q^n$ the generating function. A sketch of proof is as follows. For $n \neq 0$, one can remove the right-most arc. Then, the inner and outer part of the original diagram correspond to smaller independent configurations, of sizes n' and $n - n' - 1$. This leads to a convolutional relation for the coefficients C_n, and a polynomial one for the generating function

$$C(q) = 1 + q\, C(q)^2\,, \tag{3.1}$$

the solution matching the regularity condition $C_{-1} = 0$ being

$$C(q) = \frac{1 - \sqrt{1-4q}}{2q}\,. \tag{3.2}$$

The closed expression for the Catalan numbers is

$$C_n = \frac{1}{n+1}\binom{2n}{n}\,. \tag{3.3}$$

We say that a tree has *degree k* if all of its vertices have degree either k or 1, the latter being called *leaves* (indeed, any tree with at least one edge must have at least two leaves). The word *cubic* is used here as a synonimous of *degree 3*. We recall that, for a tree to have v_d vertices of degree d, the set of v_d's must satisfy the constraint

$$\sum_d (2-d)\, v_d = 2\,. \tag{3.4}$$

Then we will consider the problem

Problem 2. *Counting the configurations of non-intersecting cubic trees, with n vertices of degree 3 in $\mathbb{H}$ and $n+2$ leaves in $\partial\mathbb{H}$ (cfr. fig. 1, top right).*

Call $A_{1,n}$ the number of such trees, and $A_1(q)$ the generating function $A_1(q) = \sum_{n\geq 0} A_{1,n} q^n$. We have $A_{1,0} = 1$ for the tree with a single edge and two vertices of degree 1. For $n > 0$, the right-most leaf must be connected to a vertex of degree 3. Then, we can modify the drawing in a unique way so that each of the two other terminations can be seen as the right-most leaf vertex of the remaining component. Thus we have the formula

$$A_1(q) = 1 + qA_1(q)^2\,, \tag{3.5}$$

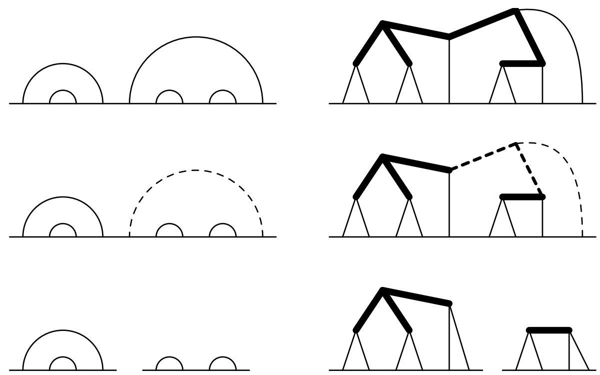


Figure 1: *On the left, a link pattern on $\mathbb{H}$ and the removal procedure which leads to the relation (3.1). On the right, a cubic tree on $\mathbb{H}$, and the removal procedure which leads to the relation (3.5).*

i.e. we deal again with Catalan numbers,

$$A_1(q) = C(q) = \frac{1-\sqrt{1-4q}}{2q}\,, \tag{3.6}$$

and $A_{1,n} = C_n$.

We will now consider a generalization to arbitrary degree k:

Problem 3. *Counting the configurations of non-intersecting trees of degree $k = h+2$, with n vertices of degree k in $\mathbb{H}$ and $hn+2$ leaves in $\partial\mathbb{H}$.*

The reasoning follows as above, with the only difference that now, under removal of the rightmost leaf and the corresponding internal vertex, we have $h+1$ remaining components, thus we have

$$A_h(q) = 1 + q\big(A_h(q)\big)^{h+1}\,, \tag{3.7}$$

which gives a generalization of Catalan numbers [44]

$$A_{h,n} = \frac{1}{hn+1}\binom{(h+1)n}{n}\,. \tag{3.8}$$

Performing a Stirling expansion of (3.8) for large n, we get

$$A_{h,n} \simeq \left(\frac{(h+1)^{h+1}}{h^h}\right)^n n^{-\frac{3}{2}}\sqrt{\frac{h+1}{2\pi h^3}}\,. \tag{3.9}$$

Conversely, equation (3.7) for the generating function for general h cannot be written in a simple form. A futher exception (besides $h=1$) is the case $h=2$, for which we have

$$A_2\Big(\frac{4x^2}{27}\Big) = \frac{3}{x}\sin\left(\frac{1}{3}\arcsin x\right)\,. \tag{3.10}$$

In Appendix A it is shown that the solution to (3.7) is a generalized hypergeometric function ${}_{h+1}F_h$, and some properties are studied. Here, we discuss a minimal set of properties which are strictly necessary in the forthcoming sections.

Defining for future convenience the combination

$$g_c(h) = 2^{-h}\frac{h^h}{(h+1)^{h+1}}\,, \tag{3.11}$$

we see that the radius of convergence for $A_h(q)$ is

$$|q| < 2^h g_c(h)\,. \tag{3.12}$$

The series has a finite value also for this value, indeed

$$A_h\big(2^h g_c(h)\big) = \frac{h+1}{h}\,, \tag{3.13}$$

and, as the series has positive summands, *a fortiori* $\left|A_h\big(2^h g_c(h) e^{i\theta}\big)\right| \leq \frac{h+1}{h}$.

Equation (3.7) has a parametric solution in algebraic form

$$\begin{cases} q^{1/h} = z = x(1-x^h)\,; \\ A_h = (1-x^h)^{-1}\,; \end{cases} \tag{3.14}$$

which is easily checked by direct substitution, and by matching the initial condition $A_{h,0} = 1$. By a simple scaling we have, for $z = x(1 - g x^h)$,

$$A_h(g z^h) = (1 - g x^h)^{-1}\,; \qquad\qquad z\big(A_h(g z^h) - 1\big) = g x^{h+1}\,. \tag{3.15}$$

The problem of counting non-intersecting trees in $\mathbb{H}$ is obviously related to the one of counting non-intersecting trees in $\mathbb{D}$ (with leaves in $\partial\mathbb{D}$). The solutions essentially do coincide (if one point on $\partial\mathbb{D}$ is marked), except for the fact that, for the case on the disk, it is natural to add a symmetry factor for the cyclic permutations of the leaves. Thus we introduce the modified quantities

$$A'_{h,n} = \frac{A_{h,n}}{hn+2}\,; \qquad\qquad A'_h(q) = \sum_{n \geq 1} q^n A'_{h,n}\,. \tag{3.16}$$

Remark that for A', differently than for A, we start summation from $n = 1$. Again the case $k = 3$, i.e. $h = 1$, gives a simple explicit formula

$$A'_1(q) = \frac{-1 + 6q - 6q^2 + (1-4q)^{\frac{3}{2}}}{12 q^2}\,. \tag{3.17}$$

More generally, the definition (3.16) implies the relation

$$z\left(A_h(g z^h) - 1\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z^2 A'_h(g z^h)\right)\,, \tag{3.18}$$

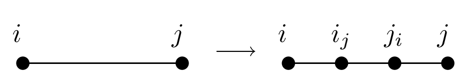
which, using $\frac{\mathrm{d}}{\mathrm{d}z} f(z(x)) = (\mathrm{d}z/\mathrm{d}x)^{-1} \frac{\mathrm{d}}{\mathrm{d}x} f(z(x))$, gives

$$z^2 A'_h(g z^h) = \frac{x^2}{2}\left(\frac{2}{h+2} g x^h - (g x^h)^2\right)\,. \tag{3.19}$$

4 Random-matrix partition function for spanning forests

Conside the set of pairs (G, F) where G is a connected graph with all vertices of degree $k = h+2$, and $F \in \mathcal{F}(G)$. With our choice for the measure, the partition function $Z(t, g, N)$ is given by (2.9). Call $\{T_\alpha\}_{\alpha=1,\ldots,K(F)}$ the components of the forest. Consider the edges $E(F) \subseteq E(G)$ as *marked*, and the remaining edges as *unmarked*. Conversely, consider *all* vertices as marked. Now the quantity $K(F)$ coincides with the number of connected marked components (possibly consisting of isolated vertices). We say that an unmarked edge is an *arc* if it connects points on the same component, and a *bridge* if it connects points on distinct components.

For each edge $(ij) \in E(G)$, call *leg-decoration* the introduction (in series) of two intermediate vertices, i_j and j_i:



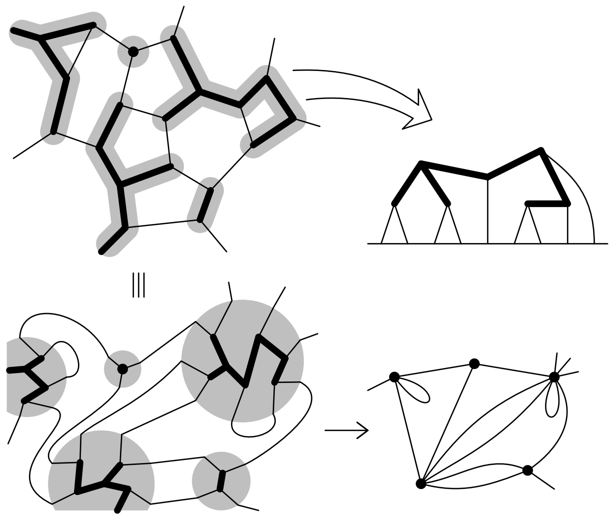


Figure 2: *On top, a portion of a typical configuration of spanning forests on a random 3-graph, and a planar cubic tree corresponding to the right-most connected component. Below, on the left, a manipulation of the drawing which highlights the shape of the "effective vertices"; on the right, the diagram of the one-matrix partition function which contains the contribution of the original configuration, obtained shrinking the effective vertices to single points, by keeping memory only of the external-leg sequence.*

Call *leg* a decorated edge of the form $(i\, i_j)$. This decoration provides a language for the mechanism of Wick contractions underlying the Random Matrix technique: the vertices coming from the expansion of the action determine the set of legs (with their cyclic ordering), while the choice of Wick contractions determines how the legs are connected.

Leg-decorate all unmarked edges. For each component T of F, we define its *border* as the set of unmarked legs incident on T. As G is actually a 2-dimensional cell complex, and any tree is homotopic to a point, the legs on the border of a tree T have an induced cyclic ordering.

Consider a component T with n vertices, together with its $hn + 2$ border legs, as a new tree T' with n vertices all of degree k, and $hn + 2$ leaves. We know from problem 3 that the number of such configurations is exactly $A_{h,n}$, and, if divided by the factor $hn + 2$ given by the cyclic symmetry, is $A'_{h,n}$.

Imagine to contract these trees $\{T'_\alpha\}$ to single vertices, without altering the cyclic ordering of the external legs. Only edges of the form $(i_j\, j_i)$ survive, and the combination of all possible n-vertex trees, with $n \geq 1$, leads to an "effective" coupling g_n for the resulting diagram, which coincides with $G \diagup F$. Recalling that we have a factor t per component, and that we introduced the proper symmetry factor in the definition of $A'_{h,n}$, we should write

$$\frac{g_n}{hn+2} = t g^n A'_{h,n} \,. \tag{4.1}$$

The combinatorics of the resulting (fat)graphs $G \diagup F$ is then suitable for resummation through Random Matrix technique. Introduce a Hermitian matrix of fields M, and consider the action

$$\begin{aligned} \mathcal{S}(M) &= \mathrm{tr}\Big(-\frac{M^2}{2} + \sum_{n \geq 1} \frac{g_n}{hn+2} M^{hn+2} \Big) \\ &= -\frac{1}{2} \mathrm{tr}\left(M^2 (1 - 2t A'_h(g M^h)) \right) \,, \end{aligned} \tag{4.2}$$

this leads to our desired generating function:

$$Z(t,g,N) = \frac{1}{N^2}\ln\int_{N\times N}\mathrm{d}M\, e^{N\mathcal{S}(M)}\,. \tag{4.3}$$

A graphical explanation of the whole procedure is presented in figure 2.

So, we still deal with a one-matrix theory, as in the case of pure-gravity, although, unfortunately the "potential" is not polynomial in matrix fields (and this causes problems, for example, in the solution methoud through the resolvent function). Again we can reduce to eigenvalues, and obtain

$$Z(t,g,N) = \frac{1}{N^2}\ln\int_N \mathrm{d}\vec{\lambda}\,\Delta^2(\vec{\lambda})\,e^{-N\sum_i V(\lambda_i)}\,; \tag{4.4}$$

$$V(\lambda) = \frac{\lambda^2}{2}(1-2tA_h'(g\lambda^h))\,. \tag{4.5}$$

where integration over values of λ is intendend inside the analiticity region for $V(\lambda)$, that is, using the result of (3.12),

$$|\lambda_i| < 2\left(\frac{g_c(h)}{g}\right)^{\frac{1}{h}}\,. \tag{4.6}$$

We can reduce the potential to a polynomial, via the proper change of variable, inspired by relation (3.19)

$$\lambda_i(x_i) = x_i(1-gx_i^h)\,; \tag{4.7}$$

such that the interesting quantities change into

$$\mathrm{d}\lambda_i = \mathrm{d}x_i(1-g(h+1)x_i^h)\,; \tag{4.8}$$

$$\begin{aligned}|\lambda_i-\lambda_j| &= |(x_i-x_j) - g(x_i^{h+1}-x_j^{h+1})| \\ &= |x_i-x_j|\cdot|1-g(x_i^h+x_i^{h-1}x_j+\cdots+x_j^h)|\,;\end{aligned} \tag{4.9}$$

$$V(x) = \frac{x^2}{2}\left(1-2g\left(1+\tfrac{t}{h+2}\right)x^h + g^2\,(1+t)\,x^{2h}\right)\,. \tag{4.10}$$

Remark how the corrections to the measure and to the Vandermonde factor combine, to give the factor

$$\widehat{\Delta}_g(\vec{x}) = \prod_{i,j}(1-g(x_i^h+x_i^{h-1}x_j+\cdots+x_j^h))\,; \tag{4.11}$$

such that now the partition function reads

$$Z = \frac{1}{N^2}\ln\int_N \mathrm{d}\vec{x}\,\Delta^2(\vec{x})\widehat{\Delta}_g(\vec{x})\,\exp\Big(-N\sum_i V(x_i)\Big)\,. \tag{4.12}$$

5 A remark on the $O(n)$ loop-gas model

Given the spherical $O(n)$ probability measure

$$\mathrm{d}\mu(\vec{\sigma}) = \frac{2}{\Omega_n}\,\delta(\sigma^2-1)\,\mathrm{d}^n\sigma\,, \tag{5.1}$$

with $\Omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, the lowest moments are given by

$$\langle\sigma^a\rangle = \langle\sigma^a\sigma^b\sigma^c\rangle = 0\,; \qquad\qquad \langle\sigma^a\sigma^b\rangle = \frac{1}{n}\delta_{ab}\,. \tag{5.2}$$

Consider a graph G in which each vertex has degree at most 3: the action

$$S_G = \sum_{(i,j)\in E(G)} \ln(1 + n\beta\, \vec{\sigma}_i \cdot \vec{\sigma}_j) \tag{5.3}$$

is such that the polynomial expansion of the integrand in the partition function involves only the moments in (5.2). More specifically, the terms of the expansion of

$$\prod_{(i,j)\in E(G)} (1 + n\beta\, \vec{\sigma}_i \cdot \vec{\sigma}_j) \tag{5.4}$$

are in correspondence with the configurations $\vec{a} \in (\{0\} \cup \{1, \dots, n\})^{E(G)}$, where an unmarked edge (ij) (that is, $a_{ij} = 0$) corresponds to a choice of the summand 1 in the expansion of factor (ij), and an edge marked with colour a (that is, $a_{ij} = a$) corresponds to the choice of summand $n\beta\, \sigma_i^a \sigma_j^a$.

Integration over the spherical measure at each vertex leaves only with configurations of marked self-avoiding loops, weighted with a factor β per marked edge. Summing over loop colourings also produces the "topological" factor n per loop [10].

The same result would have been obtained for any spherical measure such that equations (5.2) hold, i.e., up to a rescaling, for any spherical measure. We remark however that, if we choose to integrate the variables with the very special function such that not only equations (5.2) hold, but also all higher momenta vanish, the combinatorial $O(n)$-model–loop-gas correspondence extends to *generic* graphs and actions, up to the possible appearance of dimers. Indeed, assume the action has a series expansion

$$S = \sum_{\langle ij \rangle} \sum_{k\geq 1} \beta_k n^k (\vec{\sigma}_i \cdot \vec{\sigma}_j)^k\,. \tag{5.5}$$

As momenta higher than $\left\langle (\sigma^a)^2 \right\rangle$ vanish, all the coefficients β_k with $k \geq 3$ are irrelevant. We can equivalently parametrize $\beta_{1,2}$ as

$$\beta_1 = \beta\,; \qquad\qquad \beta_2 = -\tfrac{1}{2}\beta^2 + \tfrac{1}{n}\gamma\,; \tag{5.6}$$

and the integrand of the partition function is

$$e^{S(\sigma)} = \prod_{\langle ij \rangle} \left(1 + n\beta \sum_a \sigma_i^a \sigma_j^a + n\gamma \sum_a (\sigma_i^a \sigma_j^a)^2 \right) + R(\sigma)\,, \tag{5.7}$$

where the remainder term $R(\sigma)$ is a polynomial in the fields, in which each monomial has at least either a factor σ_i^3 or a factor $\sigma_i^a \sigma_i^b$ with $a \neq b$, and thus vanishes after integration, with our choice for the invariant measure. So we can consider a combinatorics of unmarked (summand 1), marked with colour a (summand $n\beta\, \sigma_i^a \sigma_j^a$) and doubly-marked with colour a (summand $n\gamma(\sigma_i^a \sigma_j^a)^2$) edges. Variable integration produces 1 if all adjacent edges are unmarked, $1/n$ if two adjacent edges are marked, and with the same colour, or one edge is doubly-marked, and 0 otherwise. So we are left with configurations of coloured self-avoiding loops and dimers, edges in the loops being weighted with a factor β, and dimers with a factor γ/n, Summing over possible colourings reproduces the "topological" factor n per loop, and rescales the weight of marked (but uncoloured) dimers to γ. The case $\gamma = 0$ in the action (5.5), on a cubic lattice, corresponds to the loop-gas problem studied in the literature, in [10] and subsequent works.

6 Connection between the spanning-forest and the $O(n)$ loop-gas model

The problem of counting configurations of self-avoiding closed loops, with a weight $\beta^L n^\ell$ given to a configuration with ℓ loops of total length L, is expected to be a combinatorial variant of

the $O(n)$ model, in particular in the case $k=3$, where the combinatorial derivation is more transparent, and for this reason has been widely studied first by Kostov, and afterwords by many others, with a large number of interesting results [32–34, 37].

Here we briefly sketck the derivation, in order to highlight the similarities with our partition function, in equation (4.12). It turns out that a stronger analogy emerges in the generalization of the loop-gas problem considered in section 5, so we will study the problem of counting configurations of self-avoiding closed loops and dimers, on random graphs of coordination $k=h+2$, with a weight $\beta^L n^\ell \gamma^d$ given to a configuration with ℓ loops, of total length L, and d dimers.

One can study the problem with a random-matrix technique, introducing $n+2$ Hermitian matrix fields, a matrix M for unmarked edges, a matrix A for the dimers, and n auxiliary matrices $\{E_\alpha\}_{\alpha=1,\dots,n}$, which mark the edges of the loops in one of n colors, in order to reproduce the "topological" factor n^ℓ. Thus we have the random-matrix partition function

$$Z_{\text{l-d gas}} = \frac{1}{N^2} \ln \int \mathrm{d}(M, A, \{E_\alpha\}) e^{N\mathcal{S}(M,A,\{E_\alpha\})}\,; \tag{6.1}$$

$$\begin{aligned}\mathcal{S} = \operatorname{tr}\Big[&-\frac{1}{2}\Big(M^2 + A^2 + \sum_\alpha E_\alpha^2\Big) + \frac{g'}{k} M^k \\ &+ \gamma^{\frac{1}{2}} g' A M^{k-1} + \sum_{\alpha,h'} \frac{\beta g'}{2} M^{h'} E_\alpha M^{h-h'} E_\alpha \Big]\,.\end{aligned} \tag{6.2}$$

If we perform the Gaussian integration of matrix A, we have

$$\mathcal{S} = \operatorname{tr}\Big(-\frac{1}{2}(M^2 + \sum_\alpha E_\alpha^2) + \frac{g'}{k} M^k + \gamma {g'}^2 M^{2k-2} + \sum_{\alpha,h'} \frac{\beta g'}{2} M^{h'} E_\alpha M^{h-h'} E_\alpha \Big)\,. \tag{6.3}$$

Consider the change of variables

$$M \to U \Lambda U^{-1}\,; \qquad\qquad E_\alpha \to U E_\alpha U^{-1}\,; \tag{6.4}$$

where the Jacobian is just the one-matrix Vandermonde determinant, as the transformation is unitary on the auxiliary matrices E_α. Now, for any pair $i \leq j$, each term $(E_\alpha)_{ij}$ appears in the action with a quadratic contribution proportional to

$$(E_\alpha)_{ij}(E_\alpha)_{ji}\Big(1 - \beta g'(\lambda_i^h + \lambda_i^{h-1}\lambda_j + \dots + \lambda_j^h)\Big) \tag{6.5}$$

and thus the Gaussian integration of E_α degrees of freedom gives a factor $\widehat{\Delta}_{\beta g'}^{-n/2}(\vec{\lambda})$

$$Z_{\text{l-d gas}} = \frac{1}{N^2} \ln \int_N \mathrm{d}\vec{x}\, \Delta^2(\vec{x}) \widehat{\Delta}_{\beta g'}^{-n/2}(\vec{x})\, \exp\Big(-\sum_i V(x_i)\Big)\,; \tag{6.6}$$

$$V(x) = \frac{1}{2}x^2 - \frac{g'}{k}x^k - \gamma {g'}^2 x^{2k-2}\,. \tag{6.7}$$

If analitic continuation in n can be performed, at least for negative integers values of n [1], in the case $n=-2$ we recover the partition function (4.12) up to constants, with the parameter correspondence

$$\begin{cases} g'/g = 1/\beta = h+2+t\,; \\ \gamma = -\frac{1}{2}\,(g'/g)^{-2}\,(1+t)\, g^{\frac{2}{k}}\,; \end{cases} \tag{6.8}$$

the only last delicate point being the fact that in section 4 we required all x_i to be in the region of analicity of the function A'_h, while in this case we require that all the factors $(1-\beta g'(h+1)x_i^h)$ are strictly positive, in order to make positive-definite the Gaussian integration of the auxiliary

[1] Cfr. [36] for a discussion on this point.

degrees of freedom (clearly, $1-a(h+1)x_i^h > 0$ for all i implies $1-a\sum_{h'} x_i^{h'} x_j^{h-h'} > 0$ for all pairs (i,j)). One should check that these requirements coincide. A one-line argument is that the quantity $\ln \widehat{\Delta}(\vec{x})$ acts as a sort of "repulsive potential" from the border of the allowed domain, thus, as it coincides in the two cases, also the borders of the two domains must coincide.

For $h=1$, the case of Nienhuis loop-gas, with no dimers, is obtained for $\gamma=0$, that is at $t=-1$. More generally, at arbitrary degree, a loop-gas model with vertex-disjoint loops and no dimers is still obtained for $t=-1$. We remark that this value of the coupling is special in the theory: the generating function of spanning forests is 'probabilistic' only for real positive values of t, by which we mean that each configuration takes a real positive weight, which can be interpreted as an (unnormalized) Gibbs measure. For t real negative, this picture does not hold anymore, in the combinatorial formulation in terms of forests. It does hold, however, for an alternate description, in terms of spanning trees only, weighted with some non-local factors related to "activities", this being the original Tutte description of the generating function. In particular, for the generic Random Cluster model we have the definition of *internal* and *external* activities, while specialization to spanning forests gives unitary weight to externally-active edges (so that it is not necessary to count them), and gives a factor $(1+t)$ per internally-active edge. So, in the interval $t \in [-1,+\infty)$, the description of the generating function in terms of trees and activities is probabilistic. Tutte's notion of activity requires a choice of a linear order on the edge set (though the generating function of the activities is, in fact, independent of this order).

Various other definitions of notions of activities exist, leading to the same Tutte generating function, in a non-obvious way. In particular, an astonishingly different notion has been recently introduced by Olivier Bernardi [38]. His notion only requires a cyclic ordering of the edges around each vertex (and, again, the resulting generating function is, in fact, independent of this choice). Not only it is interesting that, for a generic graph, this characterization strongly restricts the range of arbitrariness in the accessory ordering structure, but also, at our purposes here, that it is specially natural for graphs which are already embedded on the Riemann sphere, as is for the ensemble of "random planar graphs" 2-dimensional cell complexes arising from Random Matrix theory. In this case, the embedding naturally defines such a cyclic ordering, and no arbitrariness whatsoever is required in the Bernardi construction of Tutte polynomial (except for the choice of a single starting directed edge). So, this is a natural candidate for the construction of a probabilistic combinatorial expansion of our generating function, and the investigation of the model at $t \searrow -1$, when internal activities are forced to vanish. This combinatorial approach could shed a light on the natural conjecture that the model is critical at $t=-1$ for any graph degree, and in the universality class of Nienhuis loop-gas model.

7 Perturbative expansion: spanning trees

In the previous sections we sketched how to deal with the model at generic t with standard random-matrix techniques, even at finite N (and, for example, have access to higher-genus generating functions via loop equations).

Beside this, a purely combinatorial approach based on the results of section 3 is enough to determine the planar partition function perturbatively in t at any given order, with a relatively small effort (also higher-genus quantities could be calculated with this technique, but this is not discussed here). This approach is interesting, not only by itself, and for the emerging combinatorics, but also for a comparison with the analogue perturbative calculations on flat 2-dimensional lattices.

In this section we perform the first-order calculation, concerning spanning trees, while in sections from 8 to 12 we will describe the higher-order technique.

So, we deal with the problem of counting the pairs (G,T), with G a connected planar graph with V vertices, all of degree $k=h+2$, and T a spanning tree on G, denoted by the symbol

$T \prec G$. We have

$$Z_1(g) = \sum_T g^{V(T)} \sum_{G \succ T} \frac{1}{|\mathrm{Aut}(G/T)|} \,. \tag{7.1}$$

The graph G/T contains only one vertex, of coordination $hV + 2$. The edges form a link pattern connecting these terminations.

The combinatorics of the coefficients, together with the constraint that hV is even (the ensemble of coordination-k graphs with V vertices is empty if both V and k are odd, as there are no pairings of an odd number of legs), produces

$$Z_1(g) = \begin{cases} \displaystyle\sum_V g^V A'_{h,V} C_{(hV+2)/2} & h \text{ even;} \\ \displaystyle\sum_{V \text{ even}} g^V A'_{h,V} C_{(hV+2)/2} & h \text{ odd.} \end{cases} \tag{7.2}$$

It is worth stressing why $A'_{h,V}$ is the appropriate coefficient: indeed it forces the graph to have at least one vertex (as $A'_{h,0} = 0$ in our definition), and has an appropriate symmetry factor which accounts for the relative cyclic rotations of the link pattern and the leaves in the tree.

So, in the case of h odd, scaling the index V in the sum by a factor 2, we have

$$Z_1(g) = \sum_V g^{2V} \frac{(2V(h+1))!}{(2V)!(hV+1)!(hV+2)!} \,, \tag{7.3}$$

from which we have the asymptotics

$$Z_1(g) \sim \sum_V \left(g \frac{(h+1)^{h+1}}{(h/2)^h} \right)^{2V} V^{-4} \,, \tag{7.4}$$

allowing to obtain the critical value of the coupling

$$g_c(h) = 2^{-h} \frac{h^h}{(h+1)^{h+1}} \,, \tag{7.5}$$

and the universal exponent -4, which is in agreement with the KPZ prediction. Indeed, we have just found that the string susceptibility γ of the random-graph spanning-tree model, defined through the relation

$$Z \sim \sum_n x^n n^{-3+\gamma} \,, \tag{7.6}$$

must be $\gamma = -1$. On the other side, the central charge of the model in flat 2-dimensional space is $c = -2$. Finally, the KPZ relation consistently predicts

$$\gamma = \frac{c - 1 - \sqrt{(25-c)(1-c)}}{12} \,. \tag{7.7}$$

Furthermore, if we recall that spanning forests emerge as a limit $q \to 0$ of the Potts model, we also find that the value of the string susceptibility is in agreement with the formula of Eynard and Bonnet [39], valid for the Potts Model in the range $q \in [0, 4]$, that, in the parametrization $q = 2 - 2\cos(\pi\nu)$ reads

$$\gamma = 1 - \frac{2}{1 \pm \nu} \,. \tag{7.8}$$

From the ratio of two consecutive summands in (7.3), we deduce that the expression for $Z_1(g)$ is a generalized hypergeometric function of variable $(g/g_c)^2$. In particular, for the case of cubic planar lattices (shifting summation index from V to $V - 1$) we have the function

$$Z_1(g) = \frac{1}{12g^2} \left({}_2F_1\left(-\tfrac{3}{4}, -\tfrac{1}{4}; 2; 2^6 g^2\right) - 1 \right) \,. \tag{7.9}$$

In a similar way we can handle the case of h even. We have the formula

$$Z_1(g) = \sum_V g^V \frac{(V(h+1))!}{V!(\frac{1}{2}hV+1)!(\frac{1}{2}hV+2)!}\,, \tag{7.10}$$

from which we have the asymptotics

$$Z_1(g) \sim \sum_V \left(g\frac{(h+1)^{h+1}}{(h/2)^h} \right)^V V^{-4}\,, \tag{7.11}$$

which again gives the universal exponent -4, and the equation (7.5) for the critical value of the coupling. From the ratio of two consecutive summands in (7.10) we deduce that we deal with a generalized hypergeometric function of variable g/g_c. In particular, for the case of coordination 4 (also in this case shifting summation index from V to $V-1$) we have the function

$$Z_1(g) = \frac{1}{6g}\left({}_2F_1\left(-\tfrac{2}{3}, -\tfrac{1}{3}; 2; 3^3 g\right) - 1 \right)\,. \tag{7.12}$$

As we are interested to the limit of large volume, it is instructive to analyze the approximate expressions (7.4) and (7.11). Parametrizing $g/g_c = e^{-\epsilon}$ if h is odd, and $g/g_c = e^{-2\epsilon}$ if h is even, and determining through Stirling expansion the overall constants in (7.4) and (7.11), we get the unique expression

$$tZ_1(g) \sim t\,\kappa(h) \sum_{n\geq 1} e^{-2\epsilon n} n^{-4}\,; \qquad\qquad \kappa(h) = \begin{cases} \dfrac{\sqrt{h+1}}{2\pi h^4} & h \text{ odd,} \\ 2\dfrac{\sqrt{h+1}}{2\pi h^4} & h \text{ even.} \end{cases} \tag{7.13}$$

(The even/odd feature comes from the fact that, in the even case, the summands have a double density). The sum gives a polylogarithmic function, whose series expansion in ϵ is

$$\mathrm{Li}_4(e^{-2\epsilon}) = \zeta(4) - 2\epsilon\,\zeta(3) + \cdots + \frac{4}{3}\epsilon^3 \ln\epsilon + \cdots\,, \tag{7.14}$$

of which we highlighted the only term in the series which is not a pure monomial in ϵ, but involves the non-analyticity $\ln\epsilon$ (this non-analiticity arising, of course, in addition to the fact that the series is not convergent if $\epsilon < 0$). We shall interpret the exact values of the terms in the series as being affected by the approximation of the summands for small n, however this general feature (that the first logarithmic term comes with a power ϵ^3), as it must be driven by the large-volume limit, is well captured by the approximation, and even the corresponding numerical coefficient $4/3$ should be exact. Indeed, the (more painful) series expansion of the exact expressions (7.9) and (7.12) gives, for $h = 1$,

$$t\,\kappa(h)^{-1} Z_1(g) = \left(\frac{8192}{315} - \frac{16\sqrt{2}\pi}{3} \right) + \left(\frac{14848}{315} - \frac{32\sqrt{2}\pi}{3} \right)\epsilon + \cdots + \frac{4}{3}\epsilon^3 \ln\epsilon + \cdots\,; \tag{7.15}$$

and for $h = 2$,

$$t\,\kappa(h)^{-1} Z_1(g) = \left(\frac{729}{5} - 24\sqrt{3}\pi \right) + \left(\frac{648}{5} - 24\sqrt{3}\pi \right)\epsilon + \cdots + \frac{4}{3}\epsilon^3 \ln\epsilon + \cdots\,. \tag{7.16}$$

So we conclude that the leading non-analytic contribution to Z_1 (in powers of ϵ) is given by

$$tZ_1(g) = \left(\frac{t}{\epsilon}\right) \left[(\mathcal{O}(\epsilon),\ \text{analytic}) + \kappa(h)\frac{4}{3}\epsilon^4 \ln\epsilon\,(1 + \mathcal{O}(\epsilon)) \right]\,. \tag{7.17}$$

8 Perturbative expansion: two components

Our goal is to construct a perturbative expansion for the partition function (2.9), at every order in t, using the combinatorial results of section 3. The case of spanning trees described in section 7 was specially simple. Also the case of 2-forests is slightly special, for a number of concerns, while from the third order on we can give a general recipe (this is done in section 9).

Consider forests consisting of two trees, named as $F = (T_1, T_2)$. We have

$$Z_2(g) = \frac{1}{2} \sum_{T_1, T_2} g^{V(T_1)+V(T_2)} \sum_{G \succ F} \frac{1}{|\mathrm{Aut}(G \diagup F)|} \,, \tag{8.1}$$

where summation over G is over connected planar graphs which contain the two trees $T_{1,2}$, and no other vertices beside the ones in the trees. The factor $\frac{1}{2}$ is due to the symmetry under exchange of the two trees. Shrink the two trees $T_{1,2}$ to single vertices $v_{1,2}$, and call $\deg(v)$ the degree of vertex v. In order to have an integer number of vertices $N_{1,2}$ per tree, as $N_{1,2} = \frac{1}{h}(\deg(v_{1,2}) - 2)$, it must be $\deg(v_{1,2}) \equiv 2$ modulo h.

Now we can deal directly with the graph $G' = G \diagup F$, at the cost of a combinatorial factor for the number of "reconstructions" of the trees. The edges of G' are either bridges or arcs[2]. For $\ell \geq 1$, consider the set S_ℓ of graphs G' with ℓ bridges. On any vertex, in the cyclic ordering, two subsequent bridge terminations are divided by a (possibly empty) link pattern configuration. The exact combinatorial factor is

$$\frac{1}{\ell} A_{h,N_1} A_{h,N_2} \,, \tag{8.2}$$

plus the constraints that ℓ, N_1 and N_2 are positive integers. The symmetry reasonings leading to this result are as follows. If we can fix a "canonical" leg among the ones incident on $v_{1,2}$, we have a factor $A_{h,N}$, instead of the $A'_{h,N}$ that one should combine with a summation on all the equivalent legs. A candidate canonical leg is a leg of a bridge edge. However, we have ℓ of them, "equivalent" because of the cyclic symmetry of the definition, so we must sum over the marking of all the equivalent bridge edges, and divide accordingly by ℓ.

Thus, we have a generating function (in some parameters $z_{1,2}^{-1}$, for reasons explained in a moment)

$$\frac{1}{\ell} \left(\frac{C(z_1^{-2})}{z_1} \frac{C(z_2^{-2})}{z_2} \right)^{\ell} \,, \tag{8.3}$$

and Z_2 is recovered from this through the formal substitution $z_{1,2}^{-(hN+2)} \to A_{h,N}$, and $z_{1,2}^{-N'} \to 0$ if $N' \not\equiv 2$ modulo h.

Remark the appearence of a recurrent combination

$$q(z) := \frac{C(z^{-2})}{z} = \frac{1}{2}\Big(z - \sqrt{z^2 - 4} \Big) \tag{8.4}$$

(this is the expression for the resolvent of the Gaussian theory), having the useful property

$$q(z)^2 = z q(z) - 1 \,. \tag{8.5}$$

Summing over ℓ we are left with the function

$$Q'(z_1, z_2) = \sum_{\ell \geq 1} \frac{1}{\ell} (q(z_1) q(z_2))^{\ell} = -\ln(1 - q(z_1) q(z_2)) \,. \tag{8.6}$$

Recalling that the function $A_h(z)$ is such that

$$\oint \frac{\mathrm{d}z}{2\pi i z} \, z^2 \big(A_h(g z^h) - 1 \big) z^{-(hn+2)} = A_{h,n} (1 - \delta_{n,0}) g^n \,, \tag{8.7}$$

[2]This is said following the terminology of page 8, i.e. an edge is an *arc* if it connects points on the same component, and a *bridge* if it connects points on distinct components.

we can use contour integration to implement the formal substitution above. Define

$$a(z) = z\big(A_h(gz^h) - 1\big)\,, \tag{8.8}$$

then we have

$$Z_2(g) = \frac{1}{2} \oint \prod_{j=1,2} \frac{\mathrm{d}z_j\, a(z_j)}{2\pi i} Q'(z_1, z_2)\,. \tag{8.9}$$

We would now need to determine the leading non-analytic term in the limit $g \nearrow g_c$, i.e. the two-component analogue of equation (7.17). For some reasons (namely, the cyclic symmetry of Feynman diagrams constituted of two vertices and ℓ edges in parallel, which leads to logarithmic series instead of geometric ones), this calculation turns out to be a subtle variant of our approach for forests with $n \geq 3$ components, analysed in the following sections, and is discussed in full detail only in section 10.

Here, and up to the end of the section, we will make a digression, in which we use a different method, w.r.t. the one of contour integration in complex plane, used in the rest of the paper. This tool succeedes in determining the nature of the leading non-analytic behaviour, but fails to provide the exact value of the corresponding numerical coefficient.

We will not enter too much in the necessary mathematical background of the method (that we could call of "Lévy calculus"), however such a pedagogical introduction should appear in a future companion paper, where a number of other statistical results are derived in this way.

We go back to the understanding of Z_2 as a sum over the number ℓ of bridges, and, given ℓ, we have 2ℓ link pattern configurations intertwined with the bridge legs, cyclically along the two vertices. So we have

$$\begin{aligned} Z_2(g) = &\sum_{\ell, N_1, N_2 \geq 1} \frac{1}{2\ell} A_{h,N_1} A_{h,N_2}\ g^{N_1+N_2} \\ &\times \sum_{\nu_1,\ldots,\nu_{2\ell} \geq 0} \delta_{hN_1+2, 2(\nu_1+\cdots+\nu_\ell)+\ell}\ \delta_{hN_2+2, 2(\nu_{\ell+1}+\cdots+\nu_{2\ell})+\ell} \prod_{\alpha=1}^{2\ell} C_{\nu_\alpha}\,. \end{aligned} \tag{8.10}$$

We can easily manipulate a number of overall factors. Define the rescaled Catalan numbers as

$$\hat{C}_n = 2^{-2n-1} C_n\,, \tag{8.11}$$

such that the corresponding generating function has radius of convergence 1, and indeed these coefficients are normalized, $\sum_{n \geq 0} \hat{C}_n = 1$. Define also the rescaled coefficients $\hat{A}_{h,N}$ as

$$\hat{A}_{h,N} = \big(2^h g_c(h)\big)^n A_{h,N} \tag{8.12}$$

so that also their generating function has radius of convergence 1 (it is however normalized to $1 + \frac{1}{h}$). Through these coefficients, the leading exponential factors are clearly highlighted

$$\begin{aligned} Z_2(g) = 2^4 &\sum_{\ell, N_1, N_2 \geq 1} \frac{1}{2\ell} \hat{A}_{h,N_1} \hat{A}_{h,N_2} \left(\frac{g}{g_c}\right)^{N_1+N_2} \\ &\times \sum_{\nu_1,\ldots,\nu_{2\ell} \geq 0} \delta_{hN_1+2, 2(\nu_1+\cdots+\nu_\ell)+\ell}\ \delta_{hN_2+2, 2(\nu_{\ell+1}+\cdots+\nu_{2\ell})+\ell} \prod_{\alpha=1}^{2\ell} \hat{C}_{\nu_\alpha}\,. \end{aligned} \tag{8.13}$$

The parameter ℓ is the only variable which entangles the contributions coming from the two trees, so for a given value of ℓ we can investigate separately each term

$$Z_2(g) = \sum_{\ell \geq 1} \frac{1}{2\ell} f_\ell(g)^2\,; \tag{8.14}$$

$$f_\ell(g) = \sum_{N \geq 1} \hat{A}_{h,N} \left(\frac{g}{g_c}\right)^N \sum_{\nu_1,\ldots,\nu_\ell \geq 0} \delta_{hN+2, 2(\nu_1+\cdots+\nu_\ell)+\ell} \prod_{\alpha=1}^{\ell} \hat{C}_{\nu_\alpha}\,. \tag{8.15}$$

The quantity f_ℓ can be easily calculated exactly for $\ell = 1$ and $h = 1$, with a method analogous to the one of section 7. The result is

$$\begin{aligned} f_1(g_c e^{-\epsilon}) &= \sum_{\nu \geq 1} \hat{A}_{1,2\nu-1} \hat{C}_\nu e^{-\epsilon(2\nu-1)} = e^\epsilon \left(1 - {}_2F_1(-\tfrac{1}{4}, \tfrac{1}{4}; 2; e^{-2\epsilon})\right) \\ &= \left(1 - \frac{32\sqrt{2}}{15\pi}\right) + \cdots - \frac{1}{4\pi}\epsilon^2 \ln \epsilon + \cdots , \end{aligned} \tag{8.16}$$

where we highlighted the leading contribution, and the leading non-analytic contribution. So, the leading contribution to f_1^2 is of order 1, while the leading non-analytic contribution comes from the cross product, and is of order $\epsilon^2 \ln \epsilon$. If we factor out a term $(t/\epsilon)^2$ in $t^2 Z_2$, we have a combination of the kind

$$\left(\frac{t}{\epsilon}\right)^2 \left(A(\epsilon^2 + \cdots) + B(\epsilon^4 + \cdots) \ln \epsilon\right) , \tag{8.17}$$

analogous in form to (7.17), with the difference that the analytic term starts from a higher order.

We will now study the function $f_\ell(g)$ for arbitrary values of ℓ, and in particular in an approximation valid for large values. At this point, we make use of the core of Lévy calculus, i.e. the generalized Central Limit Theorem for variables sampled from a heavy-tailed distribution (see for example [40, sec. 3.7]). Defining "the" Lévy distribution[3]

$$\mathcal{L}_c(x) = \sqrt{\frac{c}{2\pi}} e^{-\frac{c}{2x}} x^{-\frac{3}{2}} , \tag{8.18}$$

which is a special representative of the Lévy family of alpha-stable distributions, for $\alpha = \frac{1}{2}$ and $\beta = 1$, with pseudo-variance c. We have

$$\sum_{\nu_1, \ldots, \nu_\ell \geq 0} \delta_{N, \nu_1 + \cdots + \nu_\ell} \prod_{\alpha=1}^{\ell} \hat{C}_{\nu_\alpha} \simeq \mathcal{L}_{\frac{\ell^2}{2}}(N + \mathcal{O}(\ell)) , \tag{8.19}$$

and, in particular, typical values of N are of order ℓ^2. The reason for this is that rescaled Catalan numbers have the asymptotics $\hat{C}_n \sim \frac{1}{\sqrt{4\pi}} n^{-\frac{3}{2}}$, so that the tail of the distribution on the integers $\hat{C}_n$ matches with $\mathcal{L}_{\frac{1}{2}}(x)$, and the sum of N independent variables with parameter α and pseudo-variances c_i follows an alpha-stable distribution with parameter α, and pseudo-variance c satisfying $c^\alpha = \sum_i c_i^\alpha$. A more subtle analysis (namely, a second order in Stirling approximation for the Catalan numbers) would allow to state in (8.19)

$$\mathcal{L}_{\frac{\ell^2}{2}}(N + \mathcal{O}(\ell)) = \mathcal{L}_{\frac{\ell^2}{2}}\left(N - \tfrac{7}{12}\ell + \mathcal{O}(1)\right) , \tag{8.20}$$

so, in order to control the errors deriving from the linear part in ℓ, we will write $N - b\ell$ as argument of the Lévy distribution.

At this point an issue of factors 2 coming from an even/odd feature emerges. The point is the constraint that $2(\nu_1 + \cdots + \nu_\ell) + \ell \equiv 2$ modulo h. If h is odd, this happens approximatively with a flat probability $1/h$, when N is large, regardless of ℓ. If h is even, this can never happen if ℓ is odd, and happens approximatively with a flat probability $2/h$, when N is large and ℓ is even. To be definite, we will consider the case of odd h, although it would not be difficult to treat also the other case. We get

$$\sum_{\nu_1, \ldots, \nu_\ell \geq 0} \delta_{hN+2, 2(\nu_1 + \cdots + \nu_\ell) + \ell} \prod_{\alpha=1}^{\ell} \hat{C}_{\nu_\alpha} \simeq \frac{1}{h} \mathcal{L}_{\frac{\ell^2}{2}}\left(\tfrac{h}{2}(N - b\ell)\right) . \tag{8.21}$$

[3]Some authors name this distribution after Lévy, while others use the name of Lévy distributions for the broader family of alpha-stable distributions.

Parametrizing $g = g_c e^{-\epsilon}$, and using the asymptotic behaviour of $\hat{A}_{h,N}$ (as deduced from (3.9)), one has

$$f_\ell(g) \simeq \sum_{N\geq 1} \frac{\sqrt{h+1}\,\ell}{\pi h^3} N^{-3} \exp\left(-\frac{\ell^2}{2hN} - \frac{\epsilon h}{2}(N + b\ell)\right) . \qquad (8.22)$$

We see how the integrand has a good scaling for $N \sim \ell^2$ and $\epsilon \sim \ell^{-2}$, with the correction term in b subleading (of order $\frac{1}{\ell}$). We will neglect it from now on.

Approximating the sum with an integral (again, legitimate up to corrections of relative order $\frac{1}{\ell}$), and using the known formula

$$\int_0^\infty \frac{\mathrm{d}x}{x^3} \exp\left(-ax - \tfrac{b}{4x}\right) = \frac{8a}{b} K_2(\sqrt{ab})\,, \qquad (8.23)$$

where $K_2(x)$ is the Bessel K function of index 2, we get

$$f_\ell(g) \simeq \frac{\sqrt{h+1}\,\ell}{\pi h^3} \frac{2\epsilon h^2}{\ell^2} K_2(\sqrt{\epsilon}\ell) = \epsilon \frac{2\sqrt{h+1}}{\pi h \ell} K_2(\sqrt{\epsilon}\ell)\,. \qquad (8.24)$$

Substituting into the expression (8.14) for Z_2, we have

$$Z_2(g) \simeq \epsilon^2 \frac{2(h+1)}{(\pi h)^2} \sum_{\ell \geq 1} \frac{1}{\ell^3} K_2(\sqrt{\epsilon}\ell)^2\,. \qquad (8.25)$$

In this case, the leading non-analitic behaviour in ϵ must be deduced from the characteristics of Bessel function K_2 near the origin (as the contribution at large ℓ is suppressed both by the algebraic prefactor, and by the behaviour of K_2 itself). We have

$$K_2(2z) = \left(\frac{1}{2z^2} + \cdots\right) - \ln z \left(\frac{z^2}{2} + \cdots\right)\,, \qquad (8.26)$$

where the dots stand for further terms in a series expansion. So, the leading term in the square goes like $\frac{1}{4z^4}$, while the leading non-analitic term comes from the cross product, and goes like $-z^2 \ln z\, K_2(2z)$. Using also the formula

$$\int_a^\infty \frac{\mathrm{d}x}{x} K_2(x) = \frac{K_1(a)}{a} = \frac{1}{a^2}(1 + \mathcal{O}(a^2 \ln a))\,, \qquad (8.27)$$

we get

$$\sum_{\ell\geq 1} \frac{1}{\ell^3} K_2(\sqrt{\epsilon}\ell)^2 = \left(4\zeta(7)\epsilon^{-2} + \cdots\right) - \frac{1}{2}\ln\epsilon \left(\frac{1}{4} + \cdots\right)\,, \qquad (8.28)$$

and finally

$$t^2 Z_2(g) = \left(\frac{t}{\epsilon}\right)^2 \left[\left(\mathcal{O}(\epsilon^2),\ \text{analytic}\right) - \frac{(h+1)}{(2\pi h)^2}\epsilon^4 \ln\epsilon(1 + \mathcal{O}(\epsilon))\right]. \qquad (8.29)$$

The fact that the leading non-analytic behaviour is dominated by the cross product (thus breaking the symmetry between the two trees), and that the leading contribution to the sum (8.28) comes from small values of ℓ, is a hint towards the fact that the generating function Z_2, in its limit of large graphs, is dominated by forests in which one of the two trees is much larger than the other. We will come back to this point in sections 10 and 11.

Bessel functions do not appear anymore in the paper, so there should not be confusion with an unrelated quantity, defined in the following, for which we use the letter K.

9 Perturbative expansion: higher orders

Here we show how to construct a perturbative expansion for the partition function (2.9), at every order in t, using the combinatorial results of section 3. The cases of spanning trees and spanning forests with two components, described respectively in sections 7 and 8, were special under certain aspects, while from the third order on we can give a general recipe, being a slight modification of the technique of Cauchy integral explained at the beginning of section 8 (conversely, we do not use anymore the tools of "Lévy calculus" adopted in the remaining part of that section).

At the light of the 'hardness' of the derivation for 2-forests, in comparison with the one for trees, the topic of the present section could seem an ambitious task. However, a more careful comparison of formulas (7.17) and (8.29) leads to more optimistic expectations. Indeed, from these partial results, it looks like the series $Z(g) = \sum_n t^n Z_n(g)$, for $g = g_c e^{-\epsilon}$ near to the critical value, can be written as a double series in the variables t/ϵ and ϵ. In formulas (7.17) and (8.29) we highlighted both the *tout-court* leading behaviour, and the leading behaviour among the terms containing some non-analiticity for $\epsilon \to 0$ (namely, a factor $\ln \epsilon$). Of course, the more the latter is subleading, the more a refined control on the result is necessary, in order to extract a sufficient number of terms in a full expansion. Conversely, if the latter is leading, a first-order perturbative analysis should be sufficient. From the analysis of the cases with n components, n being 1 or 2, it can be conjectured that the disturbing analytic series starts with ϵ^n, while the first non-analytic term occurs at order $\epsilon^4 \ln \epsilon$ (in agreement with the value of the string susceptibility, and the fact that $|V| \sim 1/\epsilon$). As a result, if such a conjecture did hold for all values of n, for $n \geq 4$ we would be in the easier situation in which first-order perturbative analysis suffices.

So we start the analysis of the combinatorics for forests $F = (T_1, \dots, T_n)$, with $n \geq 3$, being spanning on a graph $G \succ F$. Shrink the trees to vertices $v_1, \dots, v_n$. The graph obtained so far has been called $G' = G/F$. Imagine G' as drawn on the Riemann sphere (i.e., on the plane, plus a point at infinity). Given an arc, two regions on the Riemann sphere are naturally identified. Say that the arc is *contractible* if at least one of the two regions contains only arcs. For the remaining edges, given two edges connecting the same pair of vertices (they could be two bridges or two non-contractible arcs), again two regions on the Riemann sphere are naturally identified. Say that the edges are *multiple* if one of the two regions contains only arcs.[4] Call $G''(G')$ the graph in which contractible arcs are removed, and $G'''(G')$ the one in which both contractible arcs are removed, and multiple edges are replaced by single edges, by shrinking out the regions containing only arcs. This procedure is depicted in figure 4. By construction, G''' does not contain neither contractible arcs, nor consecutive multiple edges. It is easily seen that, at fixed n, the number of planar graphs with these characteristics is finite, and the number of edges is at most $3n-6$.[5] In a sense, the graph G''' describes the relevant part of the adjacence structure among components of the forest. We have a 'bridge' edge between two vertices if the corresponding trees are adjacent in the full graph, and a non-contractible loop incident on a vertex if the corresponding tree T has edges in $G \setminus T$ incident on vertices of T with both terminations, and which leave some other components on each of the two sides.

The removal described above corresponds to a resummation analogous to the one done for 2-forests, in determining the combination (8.6). Consider the generating function for graphs G' with n vertices

$$Z'_n = \sum_{G'} \frac{1}{|\mathrm{Aut}(G')|} \prod_{i \in V(G')} x_i^{\deg(i)} \,, \tag{9.2}$$

[4]Remark that, if $n \geq 3$, at most one of the two regions has this property. This is one of the reasons why the case $n = 2$ is special.

[5]This is a consequence of Euler formula (2.1) for connected planar graphs, i.e. with $\mathfrak{h} = 0$ and $K = 1$, $V + F - E = 2$. Furthermore, the absence of contractible arcs and of multiple edges implies that all the faces have at least 3 sides, so that $2E \geq 3F$, equality holding for triangulations. Solving w.r.t. F gives the desired relation

$$E \leq 3V - 6 \,. \tag{9.1}$$

which we will put in relation with the desired perturbative partition function

$$Z_n = \sum_{G'} \frac{1}{|\mathrm{Aut}(G')|} \prod_{i \in V(G')} A_{h, \frac{\deg(i)-2}{h}} \, g^{\frac{\deg(i)-2}{h}} \,. \tag{9.3}$$

In equation (9.2), the contribution of an edge between vertices i and j is $x_i x_j$, so we can equivalently write

$$Z'_n = \sum_{G'} \frac{1}{|\mathrm{Aut}(G')|} \prod_{(ij) \in E(G')} (x_i x_j) \,. \tag{9.4}$$

Given a vertex with a certain number of terminations of bridges and non-contractible arcs, each interval between these terminations can be occupied by contractible arcs in an arbitrary link pattern configuration. Thus, we can "dress" the edge terminations with the substitution

$$x_i \to x_i C(x_i^2) \,, \tag{9.5}$$

and then restrict the sum to all graphs G'' which do not contain any contractible arc,

$$\begin{aligned} Z'_n &= \sum_{G''} \frac{1}{|\mathrm{Aut}(G'')|} \prod_{i=1}^{n} (x_i C(x_i^2))^{\deg(i)} \\ &= \sum_{G''} \frac{1}{|\mathrm{Aut}(G'')|} \prod_{\substack{e=(i,j) \\ \in E(G')}} (x_i C(x_i^2) x_j C(x_j^2)) \,. \end{aligned} \tag{9.6}$$

Then, multiple edges can be resummed. For each ℓ-uple of multiple edges between vertices i and j, we have a contribution $(x_i C(x_i^2) x_j C(x_j^2))^\ell$. As each edge in G''' can be originated by shrinking an arbitrary number $\ell \geq 1$ of multiple edges, we can replace them with dressed single edges through the substitution

$$x_i C(x_i^2) x_j C(x_j^2) \to \frac{x_i C(x_i^2) x_j C(x_j^2)}{1 - x_i C(x_i^2) x_j C(x_j^2)} \,, \tag{9.7}$$

to be performed in the second formulation of quantity (9.6), together with a further restriction in the sum to graphs G''' which do not have multiple edges

$$Z'_n = \sum_{G'''} \frac{1}{|\mathrm{Aut}(G''')|} \prod_{\substack{e=(i,j) \\ \in E(G''')}} \frac{x_i C(x_i^2) x_j C(x_j^2)}{1 - x_i C(x_i^2) x_j C(x_j^2)} \,. \tag{9.8}$$

We claim that the formal substitution rule $x_j^{hN+2} \to A_{h,N} g^N$ if N is a positive integer, and $x_j^{N'} \to 0$ otherwise, applied to Z'_n, gives Z_n. The only delicate point is the symmetry factor pertinent to vertices, i.e. we need to prove that a canonical way exists for marking a single leg incident to any vertex, without any other symmetry factors. Label arbitrarily the vertices with integers $1, \ldots, n$. This accounts for the already included obvious factor $|\mathrm{Aut}(G''')|^{-1}$. Remark that, as G''' is connected, each vertex must have at least one incident bridge, and that a canonical spanning tree on G''' is easily constructed, e.g. the depth-first tree starting from the 'root' vertex 1.

Note that this depth-first tree has a natural orientation towards the root, so that any non-root vertex has a single outgoing edge, oriented towards his neighbour with smaller index. Also, this tree has a natural planar embedding in $\mathbb{H}$ (with the root vertex on $\partial\mathbb{H}$), and thus a natural oriented loop which encircles it (see figure 3 for an example).

For any non-root vertex i, having vertex j as a neighbour with minimum index, we will choose as "canonical" leg one of the bridge edges between i and j. We have in general $L \geq 1$ such edges. As $n \geq 3$, at least one among i and j has other neighbours, so that the L-uple of bridges is univocally splitted into m ℓ-uples of multiple edges, and in any ℓ-uple a "first"

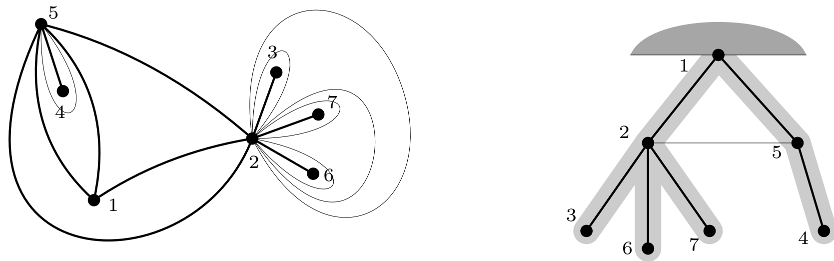


Figure 3: *On the left, a typical diagram, with labeled vertices. Bridges are drawn in bold. On the right, the corresponding depth-first tree, on the graph in which non-consecutive multiple edges are replaced by a single edge, and arcs are dropped. Remark that, in building the depth-first tree, the planar embedding of the diagram is discarded, and a new one is induced by the vertex labeling. In particular, the loop surrounding the tree is the contour of the light-gray region.*

edge in a cyclic ordering is identified (this is a difference with the $n = 2$ case, where no "first leg" can be defined). We will use as a canonical leg one of these first legs in a ℓ-uple, so we are left with $m \geq 1$ choices. We have to prove that a canonical choice among these m can be performed. At this point, the loop surrounding the tree plays a role. Indeed, the chosen pair (ij) is traversed by the loop, from i to j, exactly once, and thus a single "next" vertex k, such that (jk) follows (ij) along the loop, is identified. Because of planarity, this vertex must be contained in exactly one of the regions identified by the m multiple-edges, and thus it also identifies a canonical choice for a leg, as was to be proven. Now that we have many reference points, for the root vertex any reasonable choice makes the game. For example, one can use the other termination of the edge containing the canonical leg of the neighbour with smaller index. This completes the discussion of the involved symmetry factors.

So, we can use the technique described in formula (8.7), in order to extract Z_n from Z'_n, i.e.

$$Z_n = \prod_{j=1}^{n} \left(\oint \frac{\mathrm{d}z_j a(z_j)}{2\pi i} \right) Z'_n(\{z_j^{-1}\}) \,. \tag{9.9}$$

The quantity in (9.7), when expressed in terms of inverse parameters $\{z_j^{-1}\}$, corresponds to

$$\frac{\frac{C(z_1^{-2})}{z_1} \frac{C(z_2^{-2})}{z_2}}{1 - \frac{C(z_1^{-2})}{z_1} \frac{C(z_2^{-2})}{z_2}} = \frac{q(z_1)q(z_2)}{1 - q(z_1)q(z_2)} =: Q^-(z_1, z_2) \,, \tag{9.10}$$

where $q(z)$ has been defined in (8.4), and $Q^-(z_1, z_2)$ is a useful combination.

For future convenience, we also define

$$Q^+(z_1, z_2) := 1 + Q^-(z_1, z_2) = \frac{1}{1 - q(z_1)q(z_2)} \,, \tag{9.11}$$

and remark that these quantites can be restated as

$$Q^\pm(z_i, z_i) = \frac{1}{2} \left(\frac{z_i}{\sqrt{z_i^2 - 4}} \pm 1 \right) \;; \tag{9.12}$$

$$Q^\pm(z_i, z_j) = \frac{1}{2} \left(\frac{\sqrt{z_i^2 - 4} - \sqrt{z_j^2 - 4}}{z_i - z_j} \pm 1 \right) \;; \tag{9.13}$$

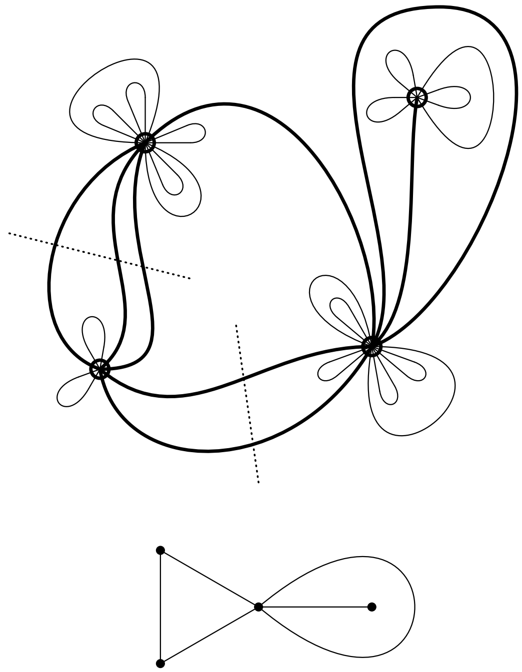


Figure 4: *On the top, a typical configuration of graph G' with $n = 4$ vertices. Contractible arcs are drawn as thin edges, and dotted lines collect multiple edges. On the bottom, the resulting graph $G'''(G')$.*

and in particular[6]

$$Q^{-}(z_i, z_i) = \frac{q(z_i)}{\sqrt{z_i^2 - 4}}\,; \tag{9.14}$$

$$Q^{-}(z_i, z_j) = -\frac{q(z_i) - q(z_j)}{z_i - z_j}\,. \tag{9.15}$$

In equation (9.9), expanded with (9.8), we can interchange integrations and summation over the graphs. Then, each term of the sum can be interpreted as a (coordinate-space) "Feynman diagram": indeed, we have the appropriate symmetry factor, one integration per vertex, with a proper "measure" factor $a(z_j)$, and a product of "propagators", or Green functions, $Q^{-}(z_i, z_j)$ corresponding to the edges of the diagram.

$$Z_n = \sum_{G'''} \frac{1}{|\mathrm{Aut}(G''')|} \oint \prod_{j=1}^{n} \frac{\mathrm{d}z_j\, a(z_j)}{2\pi i} \prod_{\substack{e=(i,j) \\ \in E(G''')}} Q^{-}(z_i, z_j)\,. \tag{9.16}$$

In section 10 we will deal with the problem of understanding the appropriate contours for such an integral, and we will prove that a valid contour exists if and only if $g \leq g_c(h)$. The integrals

[6]Equation (9.12) comes from (9.13) through a l'Hôpital limit, while (9.13), which coincides with (9.15) trivially, is related to the definition (9.10) through the use of property (8.5). Namely, calling $q = q(z)$ and $q' = q(z')$, by verifying that

$$\frac{qq'}{1 - qq'} = -\frac{q - q'}{z - z'}\,.$$

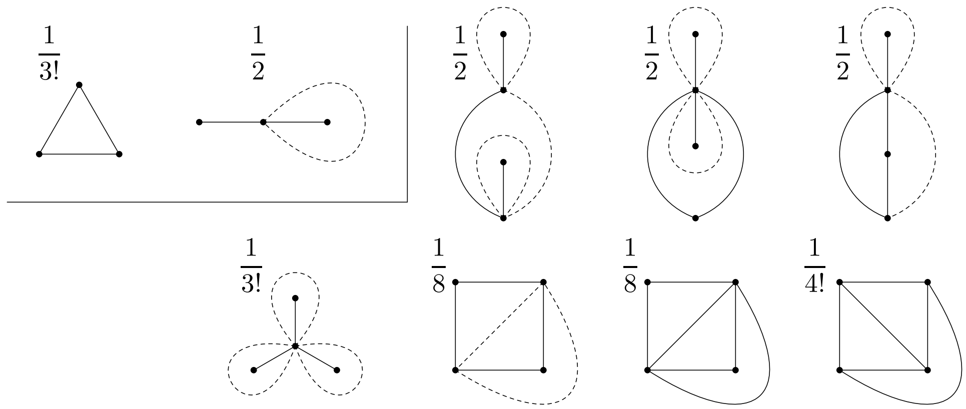


Figure 5: *Diagrams of the perturbative expansion for three-component (top-left) and four-component forests. Linear combinations, leading to the appearance of Q^+ propagators, are chosen in order to minimize the total number of diagrams. The coefficients, deriving from this combination of symmetry factors, are indicated next to each diagrams. Remark that the first non-contractible arc appears at order 3, in the right-most diagram, while the first non-consecutive multiple edges appear in some of the diagrams at order 4.*

for the third and fouth order are

$$Z_3 = \oint \prod_{j=1}^{3} \frac{\mathrm{d}z_j\, a(z_j)}{2\pi i}\ \frac{1}{6} Q^-_{12} Q^-_{23} (Q^-_{31} + 3 Q^+_{22})\,; \tag{9.17}$$

$$\begin{aligned} Z_4 = \oint \prod_{j=1}^{4} \frac{\mathrm{d}z_j\, a(z_j)}{2\pi i} & \left[\frac{1}{2} Q^-_{12} Q^-_{23} Q^-_{34} \left(\frac{1}{4} Q^-_{14} Q^+_{13} Q^+_{24} + Q^+_{22} Q^+_{33} Q^+_{23} \right) \right. \\ & \left. + \frac{1}{2} Q^-_{12} Q^-_{13} Q^-_{14} \left({Q^+_{11}}^2 Q^-_{12} + \frac{1}{3} {Q^+_{11}}^3 + Q^+_{11} Q^+_{12} Q^-_{23} + \frac{1}{12} Q^+_{23} Q^+_{34} (3 Q^-_{12} + Q^-_{24}) \right) \right]\,; \end{aligned} \tag{9.18}$$

where short notation $Q^{\pm}_{ij}$ stands for $Q^{\pm}(x_i, x_j)$, and in many cases the contributions of more diagrams are collected together, with the combination $Q^+_{ij} = 1 + Q^-_{ij}$. The related diagrams are shown in figure 5, where dashed lines denote propagators Q^+_{ij}.

Here we remark that this approach in some sense "captures" the degrees of freedom of the perturbative theory: we sum over the contribution of an infinite number of degrees of freedom (as we are summing over graphs with an arbitrary number of vertices), and we end up with a finite diagrammatics, i.e. a finite sum of finite-dimensional integrals. Remark in particular that, as expected, the diagrams in our family are all planar.

A detailed discussion on how to practically estimate the result of these integrations appears in section 10.

We remark that most of the techniques outlined in this paper for the spanning-forest model immediately generalize to variants of the model, in which more general weights are chosen for the trees

$$Z_{\text{gen.}} = \sum_{G} \frac{N^{-2\mathfrak{h}(G)}}{|\mathrm{Aut}(G)|} g^{|V(G)|} \sum_{F \prec G} \frac{|\mathrm{Aut}(G)|}{|\mathrm{Aut}(G/F)|} \prod_{T_\alpha \in F} (t\, w(T_\alpha))\,; \tag{9.19}$$

here the weights $w(T)$ for the single components (i.e. the trees) of the forest only depend on the structure of the tree (and not on the embedding into G). Our case is $w(T) = 1$. Other interesting cases are $w(T) = |T|$ (*rooted forests*, cfr. below); the generalization $w(T) = |T|^\nu$; the case $w(T) = \sum_{v \in V(T)} \xi_{\deg_T(v)}$, which is the generating function for the distribution of vertex coordinations. In all these cases, the weight $w(T)$ reflects into a modification of the generating

function $a(z)$ in (8.8), while all the formulas in which $a(z)$ appears implicitly are valid in the generic case. In particular, the case $w(T) = |T|^\nu$, for various values of ν, explores all the asymptotic behaviours of $a(z)$, "shifting" the spanning-tree string susceptibility $\gamma = -1$ to $\gamma+\nu$. This is absolutely trivial in the case of trees, as $|T| = |V(G)|$, but has substancial consequences on the measure for spanning forests at $t > 0$. In particular, for $\nu = 1$ we have a model of *rooted* spanning forests, a considerably easier variant of spanning forests, corresponding to a massive perturbation of the pure graph-Laplacian implicit in Kirchhoff Matrix-Tree theorem.

10 Evaluation of diagrams

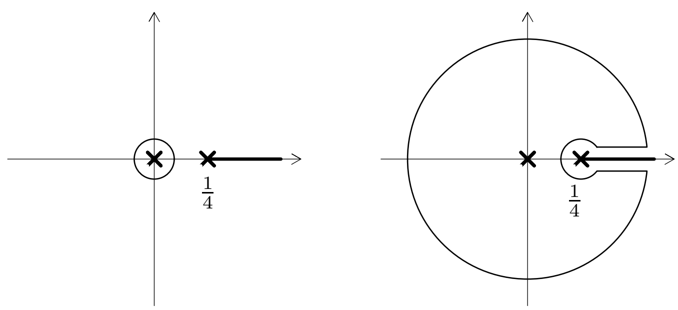
In the previous section we stated some combinatorial quantities in terms of contour integrals, which formally express certain convolutions of generating functions. In this section we discuss the analytic prescriptions on how to perform these integrations, and give a general technique to reduce our integrals to integrals of real-valued functions over real intervals.

We give a number of examples, in increasing order of difficulty, with the aim of gradually introducing a set of tools, more extensively used in the following sections. For some of these examples, also the results are of direct interest.

The easiest example of this method is still given by Catalan numbers. Suppose one wants to deduce formula (3.3) from the generating function (3.2). Then, one can perform a contour integration

$$C_n = \oint \frac{\mathrm{d}z}{2\pi i z} \, \frac{1-\sqrt{1-4z}}{2z} z^{-n} \,. \tag{10.1}$$

It is legitimate to deform the contour of integration, up to encircle the whole plane except for the cut $[\frac{1}{4}, +\infty]$:



For n sufficiently large ($n > -\frac{1}{2}$ suffices, so for any integer n), the integral on the large circle vanishes, and we are left with the integral on the two sides of the cut. For $z = x \pm i\epsilon$, with x real larger than $\frac{1}{4}$ and ϵ real positive infinitesimal, one has $\sqrt{1-4(x \pm i\epsilon)} = \mp i |\sqrt{4x-1}| + \mathcal{O}(\epsilon)$, and one can write

$$C_n = \int_{\frac{1}{4}}^{+\infty} \frac{\mathrm{d}x}{2\pi} \, \frac{\sqrt{4x-1}}{x^{n+2}} \,, \tag{10.2}$$

which gives indeed Catalan numbers.

A next case, of intermediate difficulty between the previous example and the general-diagram case, is the case of spanning trees on cubic lattices. Suppose we want to deduce the result of equation (7.9) with no use of the explicit coefficients C_n and $A'_{1,n}$, but only through the generating functions $C(x)$ and $A'_1(x)$. Start consider the case of generic h. Call n the index for the tree size, and m the one for the number of edges, then the spanning-tree partition function reads

$$Z_1(g) = \sum_{n,m} g^n A'_{h,n} C_m \delta_{2m,hn+2} \,. \tag{10.3}$$

In generating function,

$$x^2 A'_h(gx^h) = \sum_n g^n A'_{h,n} x^{hn+2} \,, \tag{10.4}$$

$$C(x^{-2}) = \sum_m C_m x^{-2m} \,, \tag{10.5}$$

and contour integration can be used in order to reproduce the delta constraint

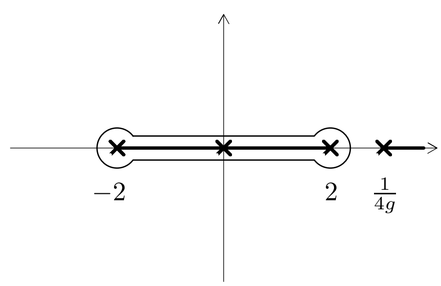
$$Z_1(g) = \oint \frac{\mathrm{d}z}{2\pi i z} \; z^2 A'_h(gz^h) C(z^{-2}) \,. \tag{10.6}$$

For the case $h = 1$, using equations (3.2) and (3.17), we have

$$z^2 A'_1(gz) = \frac{-1 + 6gz - 6(gz)^2 + (1-4gz)^{\frac{3}{2}}}{12g^2} \,, \tag{10.7}$$

$$\frac{1}{z} C(z^{-2}) = \frac{z - \sqrt{z^2-4}}{2} \,. \tag{10.8}$$

Now it is clear which prescription we should use for the contour integral: we should encircle the origin, being in the radius of convergence for both series $C(x)$ and $A'_1(x)$, thus $2 < |z| < 1/(4g)$. As expected, no contour can be found when $g > g_c = 1/8$. In terms of Cauchy integration, this prescription states that the contour should encircle the cut going from -2 to 2, and leave outside the cut going from $1/(4g)$ to infinity. If we deform the path in order to have contribution only from the $[-2,2]$ cut discontinuity, as in



we have

$$Z_1(g) = \frac{1}{12g^2} \int_{-2}^{2} \frac{\mathrm{d}x}{2\pi} \; \sqrt{4-x^2} \Big(-1 + 6gx - 6(gx)^2 + (1-4gx)^{\frac{3}{2}} \Big) \,. \tag{10.9}$$

It can be seen that this is the appropriate expression. Indeed, if we expand the generating function $A'_1(x)$, keep only even orders (because of the parity of the other factor in the integrand), and use the formula

$$\int_{-2}^{2} \frac{\mathrm{d}x}{2\pi} \; x^{2n} \sqrt{4-x^2} = C_n \tag{10.10}$$

we obtain the series coefficients (7.3).

What would have happened if we wanted to evaluate the integral by the change of variables which makes $A'_1(x)$ an analytic function? The new variable would have been $x' = x(1-gx)$. For g infinitesimal, the cut $[-2,2]$ moves infinitesimally to the two new values

$$x_+ = \frac{1-\sqrt{1-8g}}{2g} = 2 + 4g + \cdots \,, \tag{10.11}$$

$$x_- = \frac{1-\sqrt{1+8g}}{2g} = -2 + 4g + \cdots \,, \tag{10.12}$$

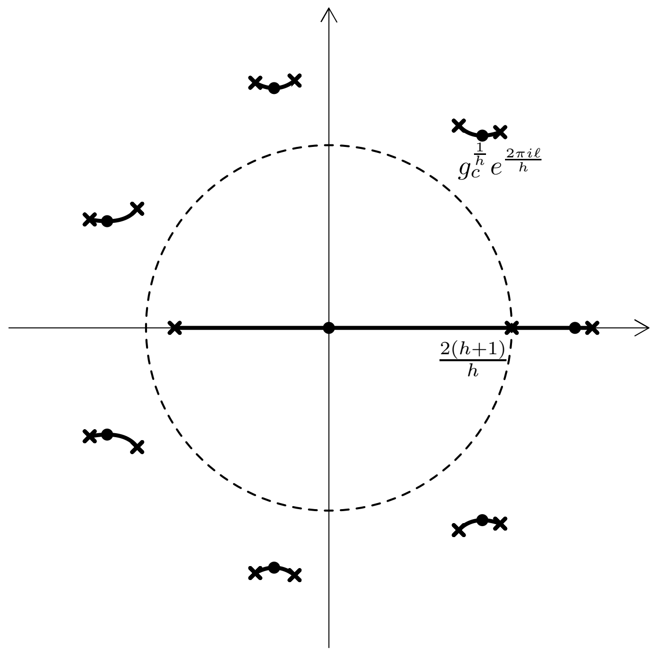


Figure 6: *The counter-images of the interval $[-2, 2]$ under the map $z = x(1-g_c x^h)$, in the complex plane for x. Here $h = 7$. The bullets correspond to the counter-images of the origin. The dashed circle denotes the largest disk centered in the origin which does not intersect any of the external cuts, and is thus the radius of convergence for the integrand corresponding to a typical diagram in the Feynman expansion. A valid contour must encircle the internal cut, staying within this disk.*

and a new cut, to be left outside the contour of integration, appears, between the values

$$x'_+ = \frac{1+\sqrt{1-8g}}{2g} = \frac{1}{g} - 2 - 4g + \cdots\,, \tag{10.13}$$

$$x'_- = \frac{1+\sqrt{1+8g}}{2g} = \frac{1}{g} + 2 - 4g + \cdots\,. \tag{10.14}$$

Above the critical value $g_c = 1/8$, the two solutions x_+ and x'_+, instead of being radially ordered, are complex conjugate, and there is no contour in which the convergence of the generating function is assured.

This phenomenology occurs also at higher values of h, with a slight even/odd difference. Indeed, one can study the counter-images of the cut $z \in [-2, 2]$ under the map $z = x(1 - gx^h)$. The value $z = 0$ has $h+1$ counter-images $x = \{0\} \cup \{g^{-\frac{1}{h}} e^{\frac{2\pi i \ell}{h}}\}_{0 \leq \ell < h}$, which 'mark' the $h+1$ counter-images of the cut if $g < g_c$ (where the critical value g_c is defined in (7.5)). However, at $g = g_c$, two real counter-images of $z = 2$, for the central cut and the cut with label $\ell = 0$, collapse, at the point $x_\star = 2\frac{h+1}{h}$. If h is even, by symmetry this also happens to two real counter-images of $z = -2$, the ones for the central cut and for $\ell = h/2$. An example for $h = 7$, at the critical coupling, is illustrated in figure 6.

Another check of the validity of equation (10.9), and a different approach to the result of equation (7.17), comes from deducing that, for $h = 1$,

$$Z_1(g) = (\mathcal{O}(1), \text{ analytic}) + \kappa(1)\frac{4}{3}\epsilon^3 \ln\epsilon\,(1 + \mathcal{O}(\epsilon))\,.$$

This is equivalent to state that

$$\frac{\partial^3}{\partial\epsilon^3} Z_1(g) = 8\kappa(1)\ln\epsilon + \mathcal{O}(1)\,. \tag{10.15}$$

Indeed, in (10.9) the parameter ϵ occurs only in the "measure" $A'_1(gz)$, and one has, for $z \sim 2$ and $g = g_c e^{-\epsilon}$,

$$\begin{aligned} \frac{\partial^3}{\partial \epsilon^3}(z^2 A'_1(gz)) &\sim -\frac{3}{8}\frac{(4gz)^3}{12g^2}(1-4gz)^{-\frac{3}{2}} + \text{more reg. terms} \\ &= -2(1-4gz)^{-\frac{3}{2}} + \text{more reg. terms.} \end{aligned} \tag{10.16}$$

The semicircle factor $\sqrt{4-z^2}$, near the singularity $z=2$, is approximated by $2\sqrt{2-z}$. Then, integrating in some window $[2-a, 2]$ near the singularity, we get

$$Z_1 \sim -\frac{2}{\pi}\int_{2-a}^{2} \mathrm{d}z\, \sqrt{2-z}\left(1-\frac{z}{2}e^{-\epsilon}\right)^{-\frac{3}{2}} = \frac{4\sqrt{2}}{\pi}\ln\epsilon + \mathcal{O}(1) = 8\kappa(1)\ln\epsilon + \mathcal{O}(1)\,, \tag{10.17}$$

as was to be shown.

Now we repeat the procedure above in the case of $Z_2(g)$, in order to determine the coefficient of the leading non-analytic term, at least for $h=1$, which was lacking in section 8. From the Lévy calculus arguments of that section, we have an expansion of the form

$$Z_2(g) = (\mathcal{O}(1), \text{analytic}) + c\,\epsilon^2 \ln\epsilon\,(1+\mathcal{O}(\epsilon))\,,$$

with c an unknown numerical constant. This is equivalent to state that

$$\frac{1}{2}\frac{\partial^2}{\partial\epsilon^2} Z_2(g) = c\ln\epsilon + \mathcal{O}(1)\,. \tag{10.18}$$

We have to go back to the expression (8.9). This expression is symmetric under exchange $z_1 \leftrightarrow z_2$, and the parameter ϵ occurs only in the "one-body measures" $a(z_1)$ and $a(z_2)$. So we can write

$$\begin{aligned} \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2} Z_2(g) &= \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\frac{1}{2}\oint\frac{\mathrm{d}z_1\, a(z_1)}{2\pi i}\oint\frac{\mathrm{d}z_2\, a(z_2)}{2\pi i}\, Q'(z_1,z_2) \\ &= \frac{1}{2}\oint\frac{\mathrm{d}z_1}{2\pi i}\left(\frac{\partial^2}{\partial\epsilon^2}a(z_1)\right)\oint\frac{\mathrm{d}z_2\, a(z_2)}{2\pi i}\, Q'(z_1,z_2) \\ &\quad + \frac{1}{2}\oint\frac{\mathrm{d}z_1}{2\pi i}\left(\frac{\partial}{\partial\epsilon}a(z_1)\right)\oint\frac{\mathrm{d}z_2}{2\pi i}\left(\frac{\partial}{\partial\epsilon}a(z_1)\right)\, Q'(z_1,z_2) \\ &=: \mathcal{A} + \mathcal{B}\,. \end{aligned} \tag{10.19}$$

We will prove in the following that the second summand, $\mathcal{B}$, does not contribute to the leading non-analytic part of Z_2. We start by calculating the leading expression corresponding to the first integral, $\mathcal{A}$. It is useful to integrate by parts w.r.t. variable z_1, in order to get a restatement which avoids logarithmic functions. This is affordable in principle also for arbitrary h, because the function $a(z)$, of which little is known, has however a simple primitive (cfr. equation (3.18)), and would give

$$\mathcal{A} = \frac{1}{2}\oint\frac{\mathrm{d}z_1}{2\pi i}\left(\frac{\partial^2}{\partial\epsilon^2}(z_1^2 A'_h(g z_1^h))\right)\frac{1}{\sqrt{z_1^2-4}}\oint\frac{\mathrm{d}z_2\, a(z_2)}{2\pi i}\, Q^-(z_1,z_2)\,. \tag{10.20}$$

The two derivatives acting on variable z_1 produce a singularity $\sim 1/z$ near $z_1 = 2$, up to more regular terms, so that, neglecting other more regular terms, we can replace $z_1 \to 2$ in all non-singular expressions, and in particular in Q^-, thus factorizing the two integrals. As discussed later (in equation (12.6)), the remaining integral in z_2 is regular for $\epsilon \to 0$, and named K_h. So we can write

$$\mathcal{A} = \frac{K_h}{2}\oint\frac{\mathrm{d}z}{2\pi i}\left(\frac{\partial^2}{\partial\epsilon^2}(z^2 A'_h(g z^h))\right)\frac{1}{\sqrt{z^2-4}} + \text{more reg. terms.} \tag{10.21}$$

Again we can deform the contour in order to take contribution from the cut, and get

$$\mathcal{A} = \frac{K_h}{2\pi} \int_{-2}^{2} \mathrm{d}z \left(\frac{\partial^2}{\partial \epsilon^2} (z^2 A'_h(gz^h)) \right) \frac{1}{\sqrt{4-z^2}} + \text{more reg. terms.} \tag{10.22}$$

and one has, for $h=1$, $z \sim 2$ and $g = g_c e^{-\epsilon}$,

$$\begin{aligned} \frac{\partial^2}{\partial \epsilon^2}(z^2 A'_1(gz)) &\sim \frac{3}{4} \frac{(4gz)^2}{12g^2} (1-4gz)^{-\frac{1}{2}} + \text{more reg. terms} \\ &= 4(1-4gz)^{-\frac{1}{2}} + \text{more reg. terms,} \end{aligned} \tag{10.23}$$

so that

$$\mathcal{A} \sim \frac{K_1}{\pi} \int_{2-a}^{2} \mathrm{d}z \, \frac{1}{\sqrt{2-z}} \left(1 - \frac{z}{2} e^{-\epsilon}\right)^{-\frac{1}{2}} = -\frac{\sqrt{2}}{\pi} K_1 \ln \epsilon + \mathcal{O}(1) \, . \tag{10.24}$$

More generally, for arbitrary h, going back to (10.21), we have

$$\frac{\partial^2}{\partial \epsilon^2}(z^2 A'_h(gz^h)) = \frac{1}{h^2} \left(z^2 \frac{\partial^2}{\partial z^2} - 3z \frac{\partial}{\partial z} + 4 \right) (z^2 A'_h(gz^h)) \, . \tag{10.25}$$

Higher derivatives lead to stronger singularities, so the leading order is due to the second-order derivative, and we can write

$$\frac{\partial^2}{\partial \epsilon^2}(z^2 A'_h(gz^h)) \simeq \frac{1}{h^2} z^2 \frac{\partial^2}{\partial z^2} (z^2 A'_h(gz^h)) = \frac{1}{h^2} z^2 \frac{\partial}{\partial z} a(z) \, . \tag{10.26}$$

Having replaced derivatives w.r.t. ϵ with derivatives w.r.t. z allows us to perform the g-dependent change of variables which simplifies the measure. In changing variables from z to x, the Jacobian in the remaining derivative and in the measure simplify, and we have in general

$$\oint \frac{\mathrm{d}z}{2\pi i} \left(\frac{\partial}{\partial z} a(z) \right) F(z) = \oint \frac{\mathrm{d}x}{2\pi i} \left(\frac{\partial}{\partial x} g x^{h+1} \right) F\big(x(1+gx^h)\big) \, , \tag{10.27}$$

while in our specific case

$$\mathcal{A} \simeq \frac{K_h}{2h^2} \oint \frac{\mathrm{d}x}{2\pi i} ((h+1) g x^h) \frac{\big(x(1+gx^h)\big)^2}{\sqrt{\big(x(1+gx^h)\big)^2 - 4}} \, . \tag{10.28}$$

We can now reduce to an integral over the cut discontinuity due to the square root at denominator. Suppose that h is odd, so that we have a single point of singularity (instead of two symmetric ones). This singularity occurs at the rightmost extremum of the central cut (cfr. figure 6), where two counter-images $x_*^{(1,2)}$ of $z=2$ coincide for $g \nearrow g_c$. Denote with index 1 the most internal solution, and with $x_* = 2\frac{h+1}{h}$ the limit for $g = g_c$.

As we are only concerned with the leading singularity, which is due to the square root at denominator $\sqrt{\big(x(1+gx^h)\big) - 2}$, in all other factors we can replace the value of g with g_c, and of x with x_*, up to less singular terms, and write

$$\mathcal{A} \simeq \frac{K_h}{2\pi h^2} \int^{x_*^{(1)}} \mathrm{d}x \cdot 1 \cdot \frac{(2)^2}{\sqrt{4}\sqrt{\big(x(1+gx^h)\big) - 2}} = \frac{K_h}{\pi h^2} \int^{x_*^{(1)}} \mathrm{d}x \frac{1}{\sqrt{x(1+gx^h) - 2}} \, . \tag{10.29}$$

Furthermore, in the polynomial of order $h+1$, $x(1+gx^h)-2$, we should highlight the two main roots $x = x_*^{(1,2)}$, and for the rest we can replace the other roots with their values for $g = g_c$. Using the fact

$$\lim_{g \to g_c} \frac{x(1+gx^h) - 2}{(x - x_*^{(1)})(x - x_*^{(2)})} = \lim_{g \to g_c} \frac{1}{2} \frac{\partial^2}{\partial x^2} \big(x(1+gx^h) - 2\big)\Big|_{x = x_*} = -\frac{h^2}{4(h+1)} \, , \tag{10.30}$$

we can write

$$\mathcal{A} \simeq \frac{K_h}{\pi h^2} \int_0 \mathrm{d}x \frac{2\sqrt{h+1}}{h} \frac{1}{\sqrt{x(\delta x_* - x)}} = \frac{2K_h\sqrt{h+1}}{\pi h^3}\big(-\ln \delta x_* + \mathcal{O}(1)\big)\,, \tag{10.31}$$

with the shortcut $\delta x_* := x_*^{(2)} - x_*^{(1)}$. As we have

$$x_*^{(2)} - x_* \sim x_* - x_*^{(1)} \sim \sqrt{\frac{2}{h(h+1)}}\sqrt{\epsilon}\,, \tag{10.32}$$

we have

$$-\ln \delta x_* = -\frac{1}{2}\ln \epsilon + \mathcal{O}(1)\,, \tag{10.33}$$

so that we get

$$\mathcal{A} = -\frac{K_h\sqrt{h+1}}{\pi h^3}\big(\ln \epsilon + \mathcal{O}(1)\big)\,. \tag{10.34}$$

This is in agreement with the special case $h = 1$ of equation (10.24).

For what concerns the integral $\mathcal{B}$, we have that the two one-body measures produce a singularity $\sim 1/\sqrt{z}$ for any of the variables $z_{1,2} \to 2$. Beside this, there is only a two-body interaction factor $Q'(z_1, z_2)$, which is regular unless *both* varables approach 2, in which case, for z_1, z_2 along the cut $[-2, 2]$ and near 2, it behaves as

$$Q'(z_1, z_2) \sim \frac{1}{2}\ln\big(\max(2 - z_1, 2 - z_2)\big) \tag{10.35}$$

so that, even taking directly $\epsilon = 0$, we get a finite result after integrating near to the only potential singularity

$$\mathcal{B} \propto \int_0^a \mathrm{d}z_1 \int_{z_1}^a \mathrm{d}z_2\, \frac{\ln z_2}{\sqrt{z_1 z_2}} = 2a(\ln a - 1)\,. \tag{10.36}$$

By collecting (10.19), (10.34) and (10.36) we get

$$Z_2(g_c e^{-\epsilon}) = \left[(\mathcal{O}(1) \text{ analytic, } \mathcal{O}(\epsilon^3)) - \frac{\sqrt{h+1}K_h}{\pi h^3}\,\epsilon^2 \ln \epsilon\right]\,. \tag{10.37}$$

The integrals arising from the full perturbative expansion are similar, although more involved, and, of course, multi-dimensional. However, as from three components on the symmetry factors are easier to handle, there is a small variety of fundamental ingredients: only the one-body measure $a(z)$ and the two-body interaction $Q^-(z, z')$ occur. Consider a diagram D with n vertices and edge-set $E(D)$. The integral is of the form

$$\mathcal{I}(D) = \prod_j \oint \frac{\mathrm{d}z_j\, a(z_j)}{2\pi i} \prod_{(ij)\in E(D)} Q^-(z_i, z_j)\,. \tag{10.38}$$

Consider the change of variables $z_j = x_j(1 - g x_j^h)$. The measure changes as

$$\mathrm{d}z\, a(z) = \mathrm{d}x\; g x^{h+1}(1 - (h+1) g x^h)\,. \tag{10.39}$$

We will keep using z_j as a shortcut to $x_j(1-gx_j^h)$, and q_j as a shortcut of $q(z_j)$, when this does not cause confusion. Recall that it is the expression $\sqrt{z_j^2 - 4}$ in $q(z_j)$ which is discontinuous at the cut, and that, in the complex plane for x, the cut to be encircled by the integration contour is the real segment $[x_-(g), x_+(g)]$, with $|x_-|, |x_+| \leq 2\frac{h+1}{h}$, and $x_+(g) \to 2\frac{h+1}{h}$ for $g \to g_c$.

There are two 'easy' contour prescriptions, in the complex planes for x_j's, which are valid for any value of $h \geq 1$, and $g \leq g_c(h)$. One is the circle of radius $2\frac{h+1}{h}$, so that one could use an "angular" parametrization $x_j = 2\frac{h+1}{h} e^{i\theta_j}$, with $\theta_j \in [0, 2\pi]$. A second one is to consider

a “cut” integral, i.e. integrate along the sides of a rectangle, centered at the origin, and of half-sides $2\frac{h+1}{h} \times \delta$, in a limit $\delta \to 0$. In this limit only the cut discontinuity survives.

In section 11 we concentrate on the angular parametrization, when trying to extract the leading behaviour for $g \to g_c$. Here we discuss briefly the way in which the cut integral should be performed. Concentrate on a given variable x_i. Through the expressions (9.14) and (9.15), the product of Q^- edge terms is, up to a prefactor not involving z_i, a function of the form

$$\left(\frac{q_i}{\sqrt{z_i^2-4}}\right)^{m_i} \prod_{j \neq i} \left(\frac{q_i - q_j}{z_i - z_j}\right)^{m_j}$$

with m's being non-negative integers. The polynomial in q_i in the numerator can be reduced to a linear function in q_i, of the form $P_0(z_i) + P_1(z_i) q_i$, by iterated use of (8.5). If m_i is even, only the square root cut discontinuity caused by the $\sqrt{z_i^2-4}$ term in q_i gives contribution to the integral, so that we have

$$\begin{aligned}
&\oint \prod_{j \neq i} \mathrm{d}z_j \cdots \oint \frac{\mathrm{d}z_i}{2\pi i} a(z_i) \frac{P_0(z_i) + P_1(z_i) q_i}{(4-z_i^2)^{\frac{m_i}{2}} \prod_{j\neq i}(z_i - z_j)^{m_j}} \\
&\quad = \oint \prod_{j \neq i} \mathrm{d}z_j \cdots \int_{-2}^{2} \frac{\mathrm{d}z_i}{2\pi} \frac{a(z_i)}{\sqrt{4-z_i^2}} \frac{P_1(z_i)}{(4-z_i^2)^{\frac{m_i-2}{2}} \prod_{j\neq i}(z_i - z_j)^{m_j}} \,.
\end{aligned} \tag{10.40}$$

If instead m_i is odd, a cut term is already included in the denominator, so only the regular terms in the numerator contributes

$$\begin{aligned}
&\oint \prod_{j \neq i} \mathrm{d}z_j \cdots \oint \frac{\mathrm{d}z_i}{2\pi i} a(z_i) \frac{P_0(z_i) + P_1(z_i) q_i}{(4-z_i^2)^{\frac{m_i}{2}} \prod_{j\neq i}(z_i - z_j)^{m_j}} \\
&\quad = \oint \prod_{j \neq i} \mathrm{d}z_j \cdots \int_{-2}^{2} \frac{\mathrm{d}z_i}{2\pi} \frac{a(z_i)}{\sqrt{4-z_i^2}} \frac{2P_0(z_i) + z_i P_1(z_i)}{(4-z_i^2)^{\frac{m_i-1}{2}} \prod_{j\neq i}(z_i - z_j)^{m_j}} \,.
\end{aligned} \tag{10.41}$$

In particular, a simple case of these integrals, in some variable z_i, is when i is adjacent to only one vertex j, through ℓ edges. In this case we have

$$\begin{aligned}
\Phi^{(\ell)}(w; g) &= \oint \frac{\mathrm{d}z}{2\pi i} a(z) \left(\frac{q(w) - q(z)}{z - w}\right)^{\ell} \\
&= \int_{-2}^{2} \frac{\mathrm{d}z_i}{2\pi} a(z_i) \frac{\sqrt{4-z_i^2}}{(z-w)^{\ell}} \sum_h \binom{\ell}{2h+1} \left(q(w) - \frac{z}{2}\right)^{\ell-2h-1} (z^2-4)^h \,.
\end{aligned} \tag{10.42}$$

The case $\ell = 1$, in the specialization $h = 1$ and $g = g_c$, gives

$$\begin{aligned}
\Phi^{(1)}(w; g_c) &= -\oint \frac{\mathrm{d}z}{2\pi i} a(z) \frac{q(z) - q(w)}{z - w} = -\int_{-2}^{2} \frac{\mathrm{d}z}{2\pi} a(z) \frac{\sqrt{4-z^2}}{z-w} \\
&= \frac{4\sqrt{2}}{3\pi}(2 - 3w) + (1 - (w-4)q(w)) - \frac{\sqrt{2}}{\pi} \frac{4-w^2}{\sqrt{w+2}} \ln \frac{\sqrt{w+2}+2}{\sqrt{w+2}-2} \,.
\end{aligned} \tag{10.43}$$

For w near to the endpoint of the cut,

$$\Phi^{(1)}(2+x; g_c) = \left(3 - \frac{16\sqrt{2}}{3\pi}\right) - 2\sqrt{x} - \frac{1}{\pi} x \ln x + \frac{4\sqrt{2}}{\pi}(2\ln 2 - 1)\, x + \mathcal{O}(x^{\frac{3}{2}}) \,. \tag{10.44}$$

A calculation similar to the one performed in (10.43) leads to $\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi^{(1)}(w; g_c e^{-\epsilon})\big|_{\epsilon=0}$, which is a long expression that we do not write here. However, we report the equivalent of (10.44)

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\Phi^{(1)}(2+x; g_c e^{-\epsilon})\Big|_{\epsilon=0} = \frac{2\sqrt{2}}{\pi} \ln x + \left(4 + \frac{4\sqrt{2}}{3\pi} - 6\sqrt{2}\ln 2\right) + \mathcal{O}(x^{\frac{1}{2}}) \,. \tag{10.45}$$

Collecting the two results of (10.44) and (10.45) we finally obtain

$$\begin{aligned}\Phi^{(1)}(2+x;g_c e^{-\epsilon}) = &\left(3-\frac{16\sqrt{2}}{3\pi}\right) - 2\sqrt{x} - \frac{1}{\pi}x\ln x + \frac{4\sqrt{2}}{\pi}(2\ln 2 - 1)\,x \\ &+ \frac{2\sqrt{2}}{\pi}\epsilon\ln x + \left(4+\frac{4\sqrt{2}}{3\pi} - 6\sqrt{2}\ln 2\right)\epsilon + \mathcal{O}(x^{\frac{1}{2}}\epsilon, x^{\frac{3}{2}})\,.\end{aligned} \tag{10.46}$$

We thus see that the limit for $w \to 2$ of $\Phi^{(1)}$ is finite. Curiously, it coincides with the one for $\Phi^{(2)}$, that is

$$\Phi^{(1)}(2;g) = \Phi^{(2)}(2;g)\,, \tag{10.47}$$

as the difference in the integrand is given by a factor (cfr. the general expression (10.42)) $\binom{2}{1}\frac{q(w)-z/2}{w-z}$, which goes to 1 identically in z, for $w \to 2$.

11 Non-analytic behaviour for $g \nearrow g_c$

In order to highlight the leading contribution to $\{Z_n\}_{n\geq 3}$ in the large-volume limit $g \nearrow g_c$, we consider the series expansion described in section 9, in "angular parametrization", that is parametrizing the x_j's and g as

$$x_j = \frac{2(h+1)}{h}e^{i\theta_j}\,; \qquad\qquad g = g_c e^{-\epsilon}\,. \tag{11.1}$$

In particular, the recurrent combination gx^h becomes $\frac{1}{h+1}e^{-\epsilon+ih\theta}$. The various relevant quantities become

$$\frac{\mathrm{d}z}{2\pi i}a(z) = \frac{\mathrm{d}\theta}{2\pi}\,\frac{4(h+1)}{h^2}\,e^{-\epsilon+i(h+2)\theta}(1-e^{-\epsilon+ih\theta}) =: \frac{\mathrm{d}\theta}{2\pi}\mu(\theta)\,; \tag{11.2}$$

$$z = 2e^{i\theta}\left(1+\tfrac{1}{h}(1-e^{-\epsilon+ih\theta})\right); \tag{11.3}$$

$$\sqrt{z^2-4} = z\sqrt{1-e^{-2i\theta}(1+\tfrac{1}{h}(1-e^{-\epsilon+ih\theta}))^{-2}}\,; \tag{11.4}$$

We will also adopt the shortcut $f(\theta_i,\theta_j)$ for $Q^-(z(\theta_i),z(\theta_j))$.

There are various potential sources of non-analiticities for $g \nearrow g_c$, due to non-regularities of the integrand for $\theta_i \to 0$ or $\theta_i - \theta_j \to 0$ for the various indices, when ϵ approaches 0.

A first singularity may come from the pole $z_i - z_j$ in the denominators of $Q^{\pm}(z_i,z_j)$. However, for generic values of z_i, this is not a true singularity, as, for $z_i \to z_j$, also the numerator vanishes with the same behaviour (this is what allowed us to determine an expression for $Q^{\pm}(z_i,z_i)$). A first source of true non-analiticity comes, in the limit $\theta_j \to 0$,[7] from combinations of the form

$$1-e^{-\epsilon+ih\theta} = \epsilon - ih\theta + \tfrac{1}{2}h^2\theta^2 + \mathcal{O}(\epsilon^2,\epsilon\theta,\theta^3)\,. \tag{11.5}$$

Consistently with the fact that we drop higher orders in θ, we may adopt this approximation only in some small window $\theta \in [-\delta,\delta]$, $\delta \ll 1$.

We expanded at second order in θ because, in the integration, only overall even monomials contribute. The measure $\mu(\theta)$ gives at leading orders, besides a factor as in (11.5),

$$e^{-\epsilon+i(h+2)\theta} = 1 + i(h+2)\theta + \mathcal{O}(\epsilon,\theta^2)\,. \tag{11.6}$$

The product of (11.5) and (11.6) gives

$$\epsilon - ih\theta + \frac{h(3h+4)}{2}\theta^2 + \mathcal{O}(\epsilon^2,\epsilon\theta,\theta^3)\,. \tag{11.7}$$

[7] Or $\theta_j \to \pi$, if h is even, however we neglect this for simplicity. It would be easy to reintroduce certain factors 2 overall at the end, in the case of even h, while the present treatment covers the case of h odd.

We will see in a moment that terms odd in θ do not play any role in this measure.

Indeed, a further source of non-analiticity for $\theta \to 0$ is the combination $z-2$, appearing as a factor in the square roots inside terms $Q^{\pm}$. In this case, a stronger cancellation, also of the terms linear in θ, occurs. We get for the combination in (11.3)

$$z = 2 + \tfrac{2}{h}\left(\epsilon + \tfrac{h(h+1)}{2}\theta^2\right) + \mathcal{O}(\epsilon^2, \epsilon\theta, \theta^3)\,, \tag{11.8}$$

and thus we can rewrite the expression in (11.4) as

$$\frac{z^2-4}{z^2} = \tfrac{2}{h}\left(\epsilon + \tfrac{h(h+1)}{2}\theta^2\right) + \mathcal{O}(\epsilon^2, \epsilon\theta, \theta^3)\,. \tag{11.9}$$

This proves that the expressions $Q^{\pm}(z_i, z_j)$, both in the case $i=j$ and $i \neq j$, are even in θ_j at leading orders, so we can drop out odd terms in the one-body measure $\mathrm{d}\theta_j \mu(\theta_j)$, when $|\theta_j| < \delta$.

This analysis holding in an interval of small θ does not mean that the remaining part of the integral is negligible, and in general this is not the case. However, we can keep control on our expressions by considering an exact subdivision of the contribution of a diagram (10.38). Indeed, in general, for a contour integration on a path γ, and two points a and b on it, we can write $\oint_\gamma \mathrm{d}z f(z) = \int_{\gamma(a\to b)} \mathrm{d}z f(z) + \int_{\gamma(b \to a)} \mathrm{d}z f(z)$. For our angular integrations, we can thus distinguish between the small-θ and large-θ behaviour by choosing to decompose the (periodic) interval $[0, 2\pi]$ into $[-\delta, \delta]$ and $[\delta, 2\pi - \delta]$, and denote by $V' \subseteq V(D)$ the set of angular variables for which we integrate in the first interval:

$$\mathcal{I}(D) = \sum_{V' \subseteq V(D)} \mathcal{I}(D; V')\,; \tag{11.10}$$

$$\mathcal{I}(D; V') = \prod_{j \in V'} \int_{-\delta}^{\delta} \frac{\mathrm{d}\theta_j}{2\pi} \prod_{j \in V(D) \smallsetminus V'} \int_{\delta}^{2\pi-\delta} \frac{\mathrm{d}\theta_j}{2\pi} \prod_j \mu(\theta_j) \prod_{(ij) \in E(D)} f(\theta_i, \theta_j)\,. \tag{11.11}$$

Remark that, while each $\mathcal{I}(D; V')$ in (11.11) is a function of δ, the sum $\mathcal{I}(D)$ in (11.10) is independent from its value. This arbitrariness will be exploited in the next paragraphs.

Both the one-body function $\mu(\theta_j)$ and the Green function $f(\theta_i, \theta_j)$ are of order 1 if the corresponding angles are larger than δ. In particular, $\mathcal{I}(D; \emptyset)$ gives a contribution of order 1, analytic in a neighbourhood of $\epsilon = 0$. Thus it should be neglected even if it were the leading summand, in a way similar to what was shown to occur for the 1- and 2-component cases.

The function $\mu(\theta_j)$ is given by (11.7) for θ_j small, and thus gives "small" factors, of which we can give a dimensional estimate

$$\int_{-\delta}^{\delta} \frac{\mathrm{d}\theta_j}{2\pi} \mu(\theta_j) \sim \delta(\epsilon + \delta^2)\,. \tag{11.12}$$

Factors $f(\theta_i, \theta_j)$ occur for non-contractible arcs, with equal indices, and for bridges, with distinct indices. In the first case, if θ is small we get the expression in (11.9) to the power $-\frac{1}{2}$ (plus a subleading contribution of order 1):

$$f(\theta_j, \theta_j) \simeq \frac{1}{\sqrt{\frac{2}{h}}} \frac{1}{2\sqrt{\epsilon + \frac{h(h+1)}{2}\theta_j^2}} \sim (\epsilon + \delta^2)^{-\frac{1}{2}}\,. \tag{11.13}$$

In the case of bridges, we have two non-trivial cases, depending if only one angle (say, θ_i) is small, or both angles are small. In the first case, one can neglect the contribution of the small angle in the combination, as it does not produce any leading singularity, and write

$$f(\theta_i, \theta_j) \simeq \frac{1}{2}\left(\frac{\sqrt{z_j^2-4}}{z_j-2} \pm 1\right) = \frac{1}{2}\left(\sqrt{\frac{z_j+2}{z_j-2}} \pm 1\right)(1 + \mathcal{O}(\sqrt{\epsilon}, \delta)) \tag{11.14}$$

where the leading expression is of order 1 by our previous assumption that $j \notin V'$.

The result above seems to suggest that, if instead both angles are small, a singular behaviour may arise, similarly to (11.13). Indeed, in this case, we have for the leading part in $f(\theta_i, \theta_j)$

$$\frac{1}{2}\frac{\sqrt{z_i^2-4}-\sqrt{z_j^2-4}}{z_i-z_j} \simeq \frac{1}{2}\frac{\sqrt{4(z_i-2)}-\sqrt{4(z_j-2)}}{(z_i-2)-(z_j-2)} = \frac{1}{\sqrt{z_i-2}+\sqrt{z_j-2}} \tag{11.15}$$

so that we obtain

$$f(\theta_i, \theta_j) \simeq \frac{1}{\sqrt{\frac{2}{h}}} \frac{1}{\sqrt{\epsilon + \frac{h(h+1)}{2}\theta_i^2} + \sqrt{\epsilon + \frac{h(h+1)}{2}\theta_j^2}} \sim (\epsilon + \delta^2)^{-\frac{1}{2}}\,. \tag{11.16}$$

Remark how the expression (11.13) is a special case of (11.16), with no need anymore of l'Hôpital limit. At the end, we get

$$\mathcal{I}(D; V') \sim \big(\delta(\epsilon+\delta^2)\big)^{|V'|}(\epsilon+\delta^2)^{-\frac{1}{2}\cdot\#\{(ij)\in E(D): i,j\in V'\}}\,. \tag{11.17}$$

The expression above is just a crude "dimensional" analysis. However, it allows us to understand which terms, at fixed n and in the double sum over D and V', are dominant in the $\epsilon \to 0$ limit. At this aim, choose $\sqrt{\epsilon} \lesssim \delta \ll 1$, and call $E'(D, V')$ the set of edges in the subgraph of D induced by V'. Then (11.17) becomes

$$\mathcal{I}(D; V') \sim (\sqrt{\epsilon})^{3|V'|-|E'|}\,. \tag{11.18}$$

We call *order* of a pair (D, V') the integer $|E'| - 3|V'|$. Pairs with maximum order give the leading contribution. After some reflection, one understand which families at fixed $|V'|$ minimize the order (besides the trivial case $|V'| = 0$, which gives order 0):

Proposition 11.1 *All and only the pairs (D, V') minimizing $|E'| - 3|V'|$ at fixed n and $\tilde{n} = |V'|$ have the following defining properties:*

- *The diagram D is a triangulation;*
- *All vertices in $V(D) \smallsetminus V'$ have degree 1.*

Remark that the converse of the second claim, that all vertices of degree 1 are in $V(D) \smallsetminus V'$, is not true. However, this case leads to terms which are strongly subleading in the fixed-n (and arbitrary $\tilde{n}$) analysis, and trivially reabsorbed in the contributions from smaller values of $\tilde{n}$. So, with an abuse of definition for the class of leading diagrams, which is justified at the light of the forthcoming equation (12.3), we will restrict our attention to diagrams in which all vertices of degree 1 are in $V(D) \smallsetminus V'$.

For these diagrams it follows easily that, for all the pairs (D, V') as above,

- The subgraph $D'' \subseteq D$ induced by V', and with loops dropped out, must consist of triangles and consecutive multiple edges;
- For each pair of consecutive multiple edges in D'', incident on i and $j \in V'$, a non-zero number of vertices not in V' must be attached to i in the cyclic order between the two bridges, and none attached to j, or vice versa. The relative portion of the original diagram must be a triagulation.
- No vertices nor edges in $D \smallsetminus D''$ exist besides the ones described above.

Cfr. figure 7 for some examples. Pairs of this form have order $2(n - 3 - |V'|)$, if $1 \le |V'| \le n$. Remark in particular how this formula is in agreement with the special case $V' = V(D)$ and D a triangulation, where $|E'| = |E(D)| = 3n - 6$.

A simple proof of Proposition 11.1 is as follows. Of course we have exactly $n - \tilde{n} > 0$ vertices in $V(D) \smallsetminus V'$. As D's are connected, each of these vertices have degree at least 1, and there exists at least one of these vertices (say, i), adjacent to at least one vertex of V' (say, j).

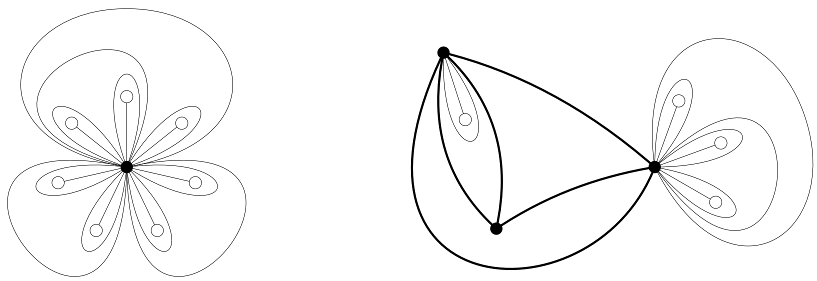


Figure 7: *Two pairs (D, V') giving the leading contribution in the set at n and $|V'|$ fixed. Vertices in V' and not in V' are denoted respectively by black and white bullets, and edges in the auxiliary graph D'' are in bold.*

In analogy with the drawings in figure 7, we call *white* and *black* the vertices respectively in $V(D) \smallsetminus V'$ and in V', so that there must exist some i white, adjacent to a black vertex j.

If i has degree larger than 1, then we will see that D cannot have maximal order, because we can build a locally modified graph D' which improves the order while remaining in the proper $(n, \tilde{n})$ class. Indeed, there is no loop in D adjacent to j and surrounding i alone. Build D' as follows: remove all edges incident on i except for (ij), add the loop on j surrounding i alone (this increases the order), then add a subset of edges between j and the other previous neighbours of i, up to make D' connected (this can only further increase the order). So we get that all white vertices neighbours to at least one black vertex, have overall degree 1. As a consequence, all white vertices are leaves, and the second part of the statement is proven.

For the first part, now consider a face of the diagram D. It visits a number of vertices, in cyclic order, possibly with repetitions. We have just seen that in this sequence we cannot have two consecutive white elements. So, if the face has 4 sides or more, there must be at least two non-adjacent terminations corresponding to black vertices. Then, we can add the corresponding edge, which is not (planar) multiple (by definition of face), and is thus allowed, and this improves the order. From this we deduce that all faces are triangles. □

Here we see how the conjecture $t^n Z_n \sim (t/\epsilon)^n (A(\epsilon^n + \cdots) + B(\epsilon^4 + \cdots) \ln \epsilon)$, deduced from the analysis of sections 7 and 8, receives new elements. The case $V' = \emptyset$ would give a contribution of the form $A(\epsilon^n + \cdots)$, while the case $|V'| = 1$, optimal for $n > 4$ (and marginally optimal for $n = 4$), would give a relative factor $\epsilon^{-(n-3-|V'|)} = \epsilon^{-(n-4)}$, compatible with the non-analytic contribution $B(\epsilon^4 + \cdots) \ln \epsilon$, and in accord with the fact that this contribution is related to the thermodynamic limit, as a non-empty set V' produces an integrand which is not blind to the $\epsilon \to 0$ singularities. At this stage of dimensional analysis, the more subtle possible presence of logarithmic factors still does not emerge.

12 Resummation of Pansy Diagrams

The defining characteristics of the dominant diagrams get simplified in the case V' consists of a single "central" vertex. All other $n-1$ vertices have degree 1, and are connected to the central vertex by a single bridge. Then, we have $2n-5$ non-contractible arcs with both terminations on the central vertex, the maximal allowed number under the constraint that there are no consecutive multiple edges, and producing a triangulation. An example is on the left of figure 7. We call these diagrams "pansy" diagrams.[8]

Note in particular that, as the adjacency matrix of these graphs is the same for a given order n, the integrals $\mathcal{I}(D, V')$ are all equal, so we just need to count the diagrams at order n,

[8]Both for the clear resemblance with the example in figure, and the fact that a triangulation on the Riemann sphere of our kind is "trilobate", and some pansy species have petals collected in three main directions.

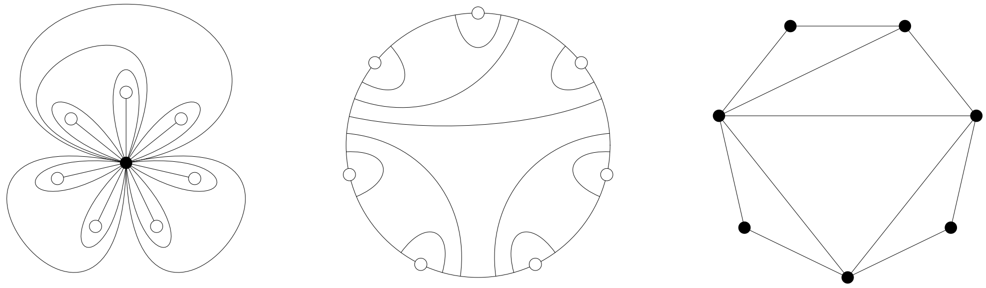


Figure 8: *On the left, a typical pansy diagram with $n = 8$. In the middle, the drawing obtained by inverting the coordinates, and compactifying the radial coordinate. On the right, the modification of this drawing which makes clear the bijection with triangulations of a polygon with $n - 1$ sides.*

and evaluate a single integral.

If one inverts the diagram (in the sense of complex-coordinate inversion $z \to 1/\bar{z}$, for a drawing in which the central vertex is at the origin), one understands that pansy diagrams are in bijection with the triangulations of regular polygons, an enumeration problem again solved by Catalan numbers (cfr. figure 8). More precisely, there are C_{n-3} triangulations of a polygon with $n-1$ vertices (contributing to n-forests), and a cyclic symmetry factor $1/(n-1)$ should be included. So at order n, from the counting of the diagrams we have a factor

$$\frac{(2n-6)!}{(n-1)!(n-3)!}\,, \tag{12.1}$$

and the integral is

$$\mathcal{I}(D,\{0\}) = \int_{-\delta}^{\delta} \frac{\mathrm{d}\theta_0}{2\pi} \prod_{j=1}^{n-1} \int_{\delta}^{2\pi-\delta} \frac{\mathrm{d}\theta_j}{2\pi} \prod_j \mu(\theta_j)\; f(\theta_0,\theta_0)^{2n-5} \prod_{j=1}^{n-1} f(\theta_0,\theta_j)\,. \tag{12.2}$$

As we know that all contributions for $\{0\} \subseteq V' \subseteq V(D)$ are subleading, we can freely include them, and get

$$\mathcal{I}(D,\{0\}) \simeq \sum_{\{0\}\subseteq V'\subseteq V(D)} \mathcal{I}(D,V') = \int_{-\delta}^{\delta} \frac{\mathrm{d}\theta_0}{2\pi} \prod_{j=1}^{n-1} \int_{0}^{2\pi} \frac{\mathrm{d}\theta_j}{2\pi} \prod_j \mu(\theta_j)\; f(\theta_0,\theta_0)^{2n-5} \prod_{j=1}^{n-1} f(\theta_0,\theta_j)\,. \tag{12.3}$$

We recognize that $n-1$ integrations are all equal, and factorized, so we can write

$$\mathcal{I}(D,\{0\}) = \int_{-\delta}^{\delta} \frac{\mathrm{d}\theta}{2\pi} \mu(\theta)\, f(\theta,\theta)^{2n-5} \left(\int_0^{2\pi} \frac{\mathrm{d}\theta'}{2\pi} \mu(\theta') f(\theta,\theta') \right)^{n-1} . \tag{12.4}$$

The integral in parenthesis, for $h = 1$, is related to the one described in equation (10.46), with

$$x(\epsilon,\theta) = z - 2 \sim \frac{2}{h} \left(\epsilon + \frac{h(h+1)}{2} \theta^2 \right) . \tag{12.5}$$

In particular, for $h = 1$, at leading order it can be replaced by the numerical constant $K_1 = 3 - \frac{16\sqrt{2}}{3\pi}$, corresponding to the limit in which both x and ϵ vanish. More generally, we expect that this limit is finite for any value of h, and call it K_h

$$K_h = \oint \frac{\mathrm{d}x}{2\pi i} g_c x^{h+1} (1-(h+1)g_c x^h)) \frac{q\big(x(1-g_c x^h)\big) - 1}{x(1-g_c x^h) - 2}\,, \tag{12.6}$$

which is the limit for $w \to 2$ and $g \to g_c$ of the quantity (10.42) in the case $\ell = 1$.

The combination x above is useful, as it also appears in $f(\theta,\theta)$

$$f(\theta,\theta) = \frac{1}{2\sqrt{x}}(1 + \mathcal{O}(\epsilon,\delta)), \tag{12.7}$$

so it is convenient to express also $\mu(\theta)$ in these terms

$$\mu(\theta) = \frac{h}{2}\frac{3h+4}{h+1}x - \frac{2h+3}{h+1}\epsilon + \mathcal{O}(\epsilon^2, \epsilon\delta, \delta^3)). \tag{12.8}$$

More precisely, it has the structure

$$\mu(\theta) = (1-\alpha)\,\epsilon + \frac{h\alpha}{2}x + \mathcal{O}(\epsilon^2, \epsilon\delta, \delta^3)), \tag{12.9}$$

with

$$\alpha = 3 + \frac{1}{h+1}, \tag{12.10}$$

and our relevant expression is

$$\mathcal{I}(D, \{0\}) = 2^{-2n+5} K_h^{n-1} \int_{-\delta}^{\delta} \frac{\mathrm{d}\theta}{2\pi} \left(\frac{h\alpha}{2}x + (1-\alpha)\,\epsilon \right) x^{-n+\frac{5}{2}}, \tag{12.11}$$

up to subleading terms. So we need to consider integrals of the form

$$W_m(\epsilon,\delta) = \int_{-\delta}^{\delta} \frac{\mathrm{d}\theta}{2\pi} x(\epsilon,\theta)^{-m-\frac{1}{2}}, \tag{12.12}$$

for $m \geq -1$. Some scale factors overall are easily extracted. Defining $\tilde{\delta} = \sqrt{\frac{h(h+1)}{2}}\delta$ we have

$$W_m(\epsilon,\delta) = \left(\frac{h}{2}\right)^m \frac{1}{\sqrt{h+1}} \int_{-\tilde{\delta}}^{\tilde{\delta}} \frac{\mathrm{d}\tau}{2\pi} (\epsilon + \tau^2)^{-m-\frac{1}{2}}. \tag{12.13}$$

The expression for Z_n, collecting also the combinatorial factor (12.1), thus reads

$$\begin{aligned} t^n Z_n &\simeq t^n \frac{(2n-6)!}{(n-1)!(n-3)!} \mathcal{I}(D, \{0\}) \\ &= t^n \frac{(2n-6)!}{(n-1)!(n-3)!} \frac{1}{2\pi} 2^{-2n+5} K_h^{n-1} \left(\frac{h\alpha}{2} W_{n-4} + (1-\alpha)\,\epsilon\, W_{n-3} \right) \\ &= \frac{t}{\pi} \frac{(2n-7)!!}{(n-1)!} \left(\frac{tK_h}{2}\right)^{n-1} \left(\frac{h\alpha}{2} W_{n-4} + (1-\alpha)\,\epsilon\, W_{n-3} \right). \end{aligned} \tag{12.14}$$

For $m = -1$ and $m = 0$ (corresponding respectively to $n = 3$ and $n = 4$) the integral W_m is not convergent in the limit $\tilde{\delta} \to \infty$, and contains a combination of logarithms that we shall discuss:

$$\int_{-\delta}^{\delta} \mathrm{d}\tau (\epsilon + \tau^2)^{\frac{1}{2}} = \delta\sqrt{\epsilon + \delta^2} + \epsilon \ln(\delta + \sqrt{\epsilon + \delta^2}) - \frac{1}{2}\epsilon \ln \epsilon\,; \tag{12.15}$$

$$\int_{-\delta}^{\delta} \mathrm{d}\tau (\epsilon + \tau^2)^{-\frac{1}{2}} = 2\ln(\delta + \sqrt{\epsilon + \delta^2}) - \ln \epsilon\,. \tag{12.16}$$

For $m > 0$ the integral is convergent. The general formula in the limit $\delta \to \infty$ is

$$\int_{-\infty}^{\infty} \mathrm{d}\tau (\epsilon + \tau^2)^{-m-\frac{1}{2}} = \epsilon^{-m} \frac{2^m (m-1)!}{(2m-1)!!}. \tag{12.17}$$

Our dimensional analysis in powers of ϵ and δ is valid for a wide range of values for δ (it suffices that $\delta \ll 1$, and $\delta \gtrsim \sqrt{\epsilon}$), so every term in the result which depends on δ in a way which is not compatible with this arbitrariness must be intended as coming from the analytic part of the integral, being this a leading term, as in the cases $n = 3$ and $n = 4$, or a subleading one, as in the case $n \geq 5$ with finite $\tilde{\delta}$ (as it is legitimate to take $\delta/\sqrt{\epsilon} \gg 1$). Thus, at the aim of understanding the non-analytic contribution, these terms can be dropped out, and we have

$$W_m^{(\mathrm{n.a.})}(\epsilon) = \frac{1}{\sqrt{h+1}} \left(\frac{\epsilon}{h}\right)^{-m} \times \begin{cases} -\frac{(-2m-1)!!}{(-m)!} \ln \epsilon & m \leq 0 \\ \frac{(m-1)!}{(2m-1)!!} & m \geq 1 \end{cases} \tag{12.18}$$

Substituting (12.18) in (12.14), again in agreement with our conjecture, we obtain the expressions for $n = 3$ and $n = 4$, that we report here together with the ones (for h odd), already derived in equations (7.17) and (10.37)

$$tZ_1(g_c e^{-\epsilon}) = \left(\frac{t}{\epsilon}\right) \left[(\mathcal{O}(\epsilon) \text{ analytic}, \mathcal{O}(\epsilon^5)) + \frac{2\sqrt{h+1}}{3\pi h^4}\, \epsilon^4 \ln \epsilon \right] ; \tag{12.19}$$

$$t^2 Z_2(g_c e^{-\epsilon}) = \left(\frac{t}{\epsilon}\right)^2 \left[(\mathcal{O}(\epsilon^2) \text{ analytic}, \mathcal{O}(\epsilon^5)) - \frac{\sqrt{h+1}\, K_h}{\pi h^3}\, \epsilon^4 \ln \epsilon \right] ; \tag{12.20}$$

$$t^3 Z_3(g_c e^{-\epsilon}) = \left(\frac{t}{\epsilon}\right)^3 \left[(\mathcal{O}(\epsilon^3) \text{ analytic}, \mathcal{O}(\epsilon^5)) + \frac{(\alpha - 2)\, K_h^2}{16\pi\sqrt{h+1}}\, \epsilon^4 \ln \epsilon \right] ; \tag{12.21}$$

$$t^4 Z_4(g_c e^{-\epsilon}) = \left(\frac{t}{\epsilon}\right)^4 \left[(\mathcal{O}(\epsilon^3) \text{ analytic}, \mathcal{O}(\epsilon^5)) - \frac{\alpha\, h\, K_h^3}{96\pi\sqrt{h+1}}\, \epsilon^4 \ln \epsilon \right] . \tag{12.22}$$

For $n \geq 5$, as the overall power of ϵ is negative, the pansy diagrams give a non-analytic quantity (in a neighbourhood of $\epsilon = 0$) already without a logarithmic factor, which indeed does not occur.

Substituting (12.18) in (12.14), we get a series for the terms with $n \geq 5$

$$\sum_{n \geq 5} t^n Z_n(g_c e^{-\epsilon}) \simeq \frac{(t/\epsilon)\, \epsilon^4}{\pi\, h^2\, \sqrt{h+1}} \sum_{n \geq 5} \left(\frac{t}{\epsilon} \frac{hK_h}{2} \right)^{n-1} \left(\frac{(n-4)!}{(n-1)!} + \frac{\alpha}{2} \frac{(n-5)!}{(n-1)!} \right) . \tag{12.23}$$

The two series are trivial, and have radius of convergence 1. The behaviour near to this point is deducible from the exact expressions

$$\sum_{n \geq 5} (1-x)^{n-1} \frac{(n-4)!}{(n-1)!} = -\frac{1}{2} x^2 \ln x + \frac{1}{12}(1-x)(1 - 5x - 2x^2) ; \tag{12.24}$$

$$\sum_{n \geq 5} (1-x)^{n-1} \frac{(n-5)!}{(n-1)!} = \frac{1}{6} x^3 \ln x + \frac{1}{36}(1-x)(2 - 7x + 11x^2) . \tag{12.25}$$

Remark in particular the leading non-analytic behaviour in the full sum (12.23), for $\epsilon \searrow \frac{thK_h}{2}$

$$\sum_{n \geq 5} t^n Z_n(g_c e^{-\epsilon}) \simeq -\frac{\epsilon^4}{\pi K_h h^3 \sqrt{h+1}} \left(1 - \frac{t}{\epsilon} \frac{hK_h}{2} \right)^2 \ln \left(1 - \frac{t}{\epsilon} \frac{hK_h}{2} \right) , \tag{12.26}$$

regardless of the precise expression (12.10) for $\alpha(h)$. We can interpret the result of this calculation as describing the curve of critical values $g_c(t; h)$ in the theory of Spanning Forests, in a neighbourhood of $t = 0$, when the limit $\langle |V(G)| \rangle \to \infty$ has been taken before $K(F) \to \infty$. We already know from elementary means the formula for $g_c(0; h)$, equation (7.5). The result above states that, in the limit described above,

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \ln g_c(t; h) \right|_{t=0} = -\frac{hK_h}{2} , \tag{12.27}$$

with K_h a numerical constant, obtained from a single (non-singular) one-dimensional contour integral, in equation (12.6), and known for $h = 1$.

The whole analysis of the pansy diagrams, besides being a remarkable exact result, shows a "topological" fact of this peculiar large-volume limit, namely that, for t, ϵ small and

$$\epsilon \searrow \epsilon_*(t) := \frac{thK_h}{2}\,, \tag{12.28}$$

the partition sum is dominated by forests whose adjacency diagram is compatible with the presence of a single gigantic tree, and many small trees. Although this statement is not quantitative, it has a clear topological reformulation: at order $K(F) = n$ fixed and in the limit $\langle |V(G)| \rangle \to \infty$, almost surely there is a single tree T which is neighbour of any other tree T', and, for each T', there are edges in $G \setminus F$ with both endpoints on T, such that the edge, together with the unique path connecting the endpoints on T, makes a cycle which encircles T'. At the same time, no pairs T', T'' of trees distinct from T are adjacent.

13 Conclusions and perspectives

The statistical mechanics of spanning forests on various graphs has two main kinds of criticality, in the "probabilistic regime" of t real non-negative, besides the trivial "high temperature" fixed point $t \to +\infty$. The point $t = 0$ corresponds to a massless theory of a scalar fermion, at the light of Kirchhoff Matrix-Tree theorem. Furthermore, at some critical value $t^\star$, a *percolation* transition may occur, i.e. for values $t < t^\star$ there exist trees which occupy a fraction of order 1 of the volume (*gigantic* compents), while for $t > t^\star$ all trees have a characteristic size, depending on t alone and not scaling with the volume. This is the specialization to forests of a feature holding more generally for the probabilistic sector of the Random Cluster (Potts) model. It is shown numerically in three, four and five dimensions [41], and analytically in the "infinite-dimensional" limit of fully connected graphs [42] that $t^\star$ is finite and non-zero in these cases, while it is expected that the arising asymptotic freedom for the model of spanning forests in two Euclidean dimensions is due to the fact that the "Kirchhoff" criticality and the percolation criticality do coincide exactly for $d = 2$.

Indeed, it was also at the aim of understanding rigorously this set of conjectures, that we performed the study of the model on Random Planar Graphs, with the aim of combining the results with KPZ tools.

We have however to face a problem in the interpretation of the results of the previous section. As we said, we have performed a double limit of $\langle |V(G)| \rangle \to \infty$ and $K(F) \to \infty$, where the first one has been performed before the second one. On the contrary, at least on Euclidean lattices, for any finite t, we expect a macroscopic number of components, i.e. $\left\langle \frac{K(F)}{|V(G)|} \right\rangle = \mathcal{O}(1)$, and, in order to have a better understanding on the behaviour of the system, we would like to control the double limit $\langle |V(G)| \rangle, K(F) \to \infty$ with arbitrary scaling.

We have seen how, in the Feynman expansion, the accessory parameter V' has a deeper meaning than it was legitimate to expect: when, for a given diagram, V' is chosen in order to maximize the contribution, we have that vertices respectively in V' and not, correspond to gigantic and small trees. So we expect that a control on the microcanonical ensemble for $\tilde{n} := |V'|$ would help at the aim of understanding the various limits. This is in a sense generalizing the approach of the previous section, where we stated that $\tilde{n} = 0$ must give an analytic contribution, coming from graphs of small size, and we analyzed exactly the leading contribution to $\tilde{n} = 1$ for $g \to g_c$.

Proposition 11.1 already identifies the class of leading diagrams at both n and $\tilde{n}$ fixed. We plan in the next future to attack the problem of re-summation of diagrams in n at further values of $\tilde{n}$.

A Generating functions for k-trees and Hypergeometric functions

We recall the definitions (3.8, 3.16)

$$A_{h,n} = \frac{((h+1)n)!}{n!(hn+1)!}\,; \qquad A_h(\omega) = \sum_{n\geq 0} \omega^n A_{h,n}\,;$$

$$A'_{h,n} = \frac{((h+1)n)!}{n!(hn+2)!}\,; \qquad A'_h(\omega) = \sum_{n\geq 1} \omega^n A'_{h,n}\,.$$

From the ratio of two consecutive summands

$$\frac{A_{h,n+1}}{A_{h,n}} = \frac{(h+1)^{h+1}}{h^h} \frac{\left(n+\frac{h}{h+1}\right)\cdots\left(n+\frac{1}{h+1}\right)}{\left(n+\frac{h+1}{h}\right)\cdots\left(n+\frac{2}{h}\right)} \tag{A.1}$$

and

$$\frac{A'_{h,n+1}}{A'_{h,n}} = \frac{(h+1)^{h+1}}{h^h} \frac{\left(n+\frac{h}{h+1}\right)\cdots\left(n+\frac{1}{h+1}\right)}{\left(n+\frac{h+2}{h}\right)\cdots\left(n+\frac{3}{h}\right)} \tag{A.2}$$

and the definition of generalized Hypergeometric function [43]

$$ {}_pF_q(\mathbf{a};\mathbf{b};\omega) := \sum_{n=0}^{\infty} \alpha_n \omega^n\,; \tag{A.3}$$

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{(n+a_1)\cdots(n+a_p)}{(n+b_1)\cdots(n+b_q)(n+1)}\,; \tag{A.4}$$

$$\alpha_0 = 1\,; \tag{A.5}$$

we have

$$A_h(\omega) = {}_{h+1}F_h(\mathbf{a}_h;\mathbf{b}_h;c_h\omega)\,; \tag{A.6a}$$

$$\mathbf{a}_h = \left(\tfrac{1}{h+1},\ldots,\tfrac{h}{h+1}\right)\,; \tag{A.6b}$$

$$\mathbf{b}_h = \left(\tfrac{2}{h},\ldots,\tfrac{h-1}{h},\tfrac{h+1}{h}\right)\,; \tag{A.6c}$$

$$c_h = \frac{(h+1)^{h+1}}{h^h}\,; \tag{A.6d}$$

and

$$A'_h(\omega) = \tfrac{1}{2}\left({}_{h+1}F_h(\mathbf{a}_h;\mathbf{b'}_h;c_h\omega)-1\right)\,; \tag{A.7a}$$

$$\mathbf{b'}_h = \left(\tfrac{3}{h},\ldots,\tfrac{h-1}{h},\tfrac{h+1}{h},\tfrac{h+2}{h}\right)\,. \tag{A.7b}$$

The parameters of functions A_h and A'_h have all but one entry in common, the last one differing by 1, so they are *contiguous* as hypergeometric functions (cfr. [43], par. 2.2.1), and this accounts for the simple relation (3.18).

The hypergeometric function corresponding to $A_h(\omega)$ has been already studied by M.L. Glasser[9]. In particular, the relation (25) in his communication coincides with our parametric solution (3.14), after that $A_h(\omega)$ has been identified with the hypergeometric function in (A.6).

The hypergeometric functions ${}_pF_q$ with $p = q+1$ and non-integer parameters do not have poles and essential singularities. They have a branch-cut discontinuity between $z^* = 1$ and the point at infinity. Accounting for the rescaling constant c_h, we have $\omega^* = h^h/(h+1)^{h+1} = 2^h g_c(h)$ (cfr. definition (7.5)), and in particular $\omega^* = 1/4$ for $h = 1$, as clear from the explicit expressions (3.5) and (3.17).

[9] M.L. Glasser, pers. comm. to the Wolfram Mathworld community, Sept. 26, 2003 `http://mathworld.wolfram.com/HypergeometricFunction.html` eq. 25.

B Comparison with random spanning trees

The results above for the ensemble of random planar lattice can be compared with small effort with the case of random lattices, regardless to the genus. In this case, the only difference is that the Catalan numbers, deriving from the combinatorics of planar matchings, must be replaced with the number of arbitrary matchings of $2n$ points, which are $(2n-1)!!$. Thus, provided that hV is even, we have

$$Z_1(g) = \sum_V g^V A'_{h,V}(hV+1)!! \,. \tag{B.1}$$

Again we treat separately the two cases of h odd or even. In the case of h odd we have

$$Z_1(g) = \sum_V g^{2V} \frac{(2V(h+1))!}{(2V)!(hV+1)!\, 2^{hV+1}} \,, \tag{B.2}$$

from which we have the asymptotics

$$Z_1(g) \sim \sum_V (hV)! \left(g \frac{(h+1)^{h+1}}{(h^2/2)^{h/2}} \right)^{2V} V^{-2} \,, \tag{B.3}$$

while in the case of h even we have the formula

$$Z_1(g) = \sum_V g^V \frac{(V(h+1))!}{V!(\frac{1}{2}hV+1)!\, 2^{hV/2+1}} \,, \tag{B.4}$$

from which we have the asymptotics

$$Z_1(g) \sim \sum_V (\tfrac{1}{2}hV)! \left(g \frac{(h+1)^{h+1}}{(h^2/2)^{h/2}} \right)^{V} V^{-2} \,. \tag{B.5}$$

Remark the expected "entropic catastrophe", due to the super-exponential number of random lattices. As a consequence, the related hypergeometric function is a ${}_qF_p$, with $q - p > 1$, contrarily to what happens in the planar case, in which one deals with ${}_{p+1}F_p$ functions, which have a finite radius of convergence.

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